Modal Logic

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A **modal deduction** is a finite sequence of formulas $\langle \alpha_1, \ldots, \alpha_n \rangle$ where for each $i \leq n$ either

1. α_i is a tautology
2. α_i is a substitution instance of $\Box(p \to q) \to (\Box p \to \Box q)$
3. α_i is of the form $\Box \alpha_j$ for some $j < i$
4. α_i follows by modus ponens from earlier formulas (i.e., there is $j, k < i$ such that α_k is of the form $\alpha_j \to \alpha_i$).

We write $\vdash_{\mathbf{K}} \varphi$ if there is a deduction containing φ.

Soundness Theorem: For all Γ and formulas φ, if $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$.

Completeness Theorem: For all Γ and formulas φ, if $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.
Weak Completeness Theorem: For all formulas φ, if $\models \varphi$ then $\vdash \varphi$.

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Proving Completeness

- Let **K** denote the minimal modal logic and $\vdash \varphi$ mean φ is derivable in **K**. If Γ is a set of formulas, we write $\Gamma \vdash \varphi$ if $\vdash (\psi_1 \wedge \cdots \wedge \psi_k) \to \varphi$ for some finite set $\psi_1, \ldots, \psi_k \in \Gamma$.

- Let Γ be a set of formulas. We write $\Gamma \models \varphi$ provided for all frames $\mathcal{F}$ for all models $\mathcal{M}$ based on $\mathcal{F}$ and all states w in $\mathcal{M}$, $\mathcal{M}, w \models \Gamma$ then $\mathcal{M}, w \models \varphi$.

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- A set of formulas Γ is **consistent** provided $\Gamma \not\vdash \bot$.

Γ is a **maximally consistent set** if Γ is consistent and for each $\varphi \in \mathcal{L}$ either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$. Alternatively, Γ is consistent and every Γ' such that $\Gamma \subsetneq \Gamma'$ is inconsistent (i.e., every proper superset of Γ is inconsistent).

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Suppose that Γ is a maximally consistent set. Then,

1. If $\vdash \varphi$ then $\varphi \in \Gamma$
2. If $\varphi \rightarrow \psi \in \Gamma$ and $\varphi \in \Gamma$ then $\psi \in \Gamma$
3. $\neg\varphi \in \Gamma$ iff $\varphi \notin \Gamma$
4. $\varphi \wedge \psi \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$
5. $\varphi \vee \psi \in \Gamma$ iff $\varphi \in \Gamma$ or $\psi \in \Gamma$

Lemma (Lindenbaum's Lemma)

For each consistent set Γ, *there is a maximally consistent set* Γ' *such that* $\Gamma \subseteq \Gamma'$. *In other words, every consistent set* Γ *can be extended to a maximally consistent set.*

Definition (Canonical Model)

The canonical model for **K** is the model $\mathcal{M}^c = \langle W^c, R^c, V^c \rangle$ where

- $W^c = \{\Gamma \mid \Gamma \text{ is a maximally consistent set}\}$
- $\Gamma R^c \Delta$ iff $\Gamma^\Box = \{\varphi \mid \Box\varphi \in \Gamma\} \subseteq \Delta$
- $V^c(p) = \{\Gamma \mid p \in \Gamma\}$

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Theorem (Strong Completeness)

If $\Gamma \models \varphi$ *then* $\Gamma \vdash \varphi$

Modal Logics

PC: All propositional tautologies

N: The rule of necessitation: $\dfrac{\varphi}{\Box\varphi}$

Some Axioms

K	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
D	$\Box\varphi \to \Diamond\varphi$
T	$\Box\varphi \to \varphi$
4	$\Box\varphi \to \Box\Box\varphi$
5	$\neg\Box\varphi \to \Box\neg\Box\varphi$
L	$\Box(\Box\varphi \to \varphi) \to \Box\varphi$

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Some Normal Modal Logics

K	$K + PC + N$
T	$K + T + PC + N$
K4	$K + 4 + PC + N$
S4	$K + T + 4 + PC + N$
S5	$K + T + 4 + 5 + PC + N$
KD45	$K + D + 4 + 5 + PC + N$
GL	$K + L + PC + N$

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- Is every consistent normal modal logic strongly complete with respect to some class of frame? No: **GL** is not strongly complete, but it is weakly complete.

- Is every consistent normal modal logic strongly complete or weakly complete with respect to some class of frame? No: There are consistent normal modal logics that are not complete with respect to any class of frame.