Modal Logic

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Modal Proof Theory

✓ Natural deduction: $\Gamma \vdash^{nd}_{\mathbf{K}} \varphi$ means that there is a natural deduction proof where the last line is φ where φ is not in the scope of a subproof and all the assumptions in the proof are from Γ.

✓ Sequents: $\Gamma \vdash^{s}_{\mathbf{K}} \varphi$ means that there is a proof of the sequent $\Gamma \Rightarrow \varphi$ where each end point (called a **leaf**) is an axiom.

3. Hilbert systems

A **modal deduction** is a finite sequence of formulas $\langle \alpha_1, \ldots, \alpha_n \rangle$ where for each $i \leq n$ either

1. α_i is a tautology
2. α_i is a substitution instance of $\Box(p \to q) \to (\Box p \to \Box q)$
3. α_i is of the form $\Box\alpha_j$ for some $j < i$
4. α_i follows by modus ponens from earlier formulas (i.e., there is $j, k < i$ such that α_k is of the form $\alpha_j \to \alpha_i$).

We write $\vdash_{\mathbf{K}} \varphi$ if there is a deduction containing φ.

$$\vdash_{\mathbf{K}} \Box(p \wedge q) \to (\Box p \wedge \Box q)$$

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1.	$p \wedge q \to p$	tautology
2.	$\Box((p \wedge q) \to p)$	Necessitation 1
3.	$\Box((p \wedge q) \to p) \to (\Box(p \wedge q) \to \Box p)$	Substitution instance of K
4.	$\Box(p \wedge q) \to \Box p$	MP 2,3
5.	$p \wedge q \to q$	tautology
6.	$\Box((p \wedge q) \to q)$	Necessitation 5
7.	$\Box((p \wedge q) \to p) \to (\Box(p \wedge q) \to \Box q)$	Substitution instance of K
8.	$\Box(p \wedge q) \to \Box q$	MP 5,6
9.	$(a \to b) \to ((a \to c) \to (a \to (b \wedge c)))$	tautology
		$a := \Box(p \wedge q), b := \Box p, c := \Box q$
10.	$(a \to c) \to (a \to (b \wedge c))$	MP 4,9
11.	$\Box(p \wedge q) \to \Box p \wedge \Box q$	MP 8,10

Let Γ be a set of modal formulas. A **modal deduction of φ from Γ**, denoted $\Gamma \vdash_{\mathbf{K}} \varphi$ is a finite sequence of formulas $\langle \alpha_1, \ldots, \alpha_n \rangle$ where for each $i \leq n$ either

1. α_i is a tautology
2. $\alpha_i \in \Gamma$
3. α_i is a substitution instance of $\Box(p \to q) \to (\Box p \to \Box q)$
4. α_i is of the form $\Box \alpha_j$ for some $j < i$ and $\vdash_{\mathbf{K}} \alpha_j$
5. α_i follows by modus ponens from earlier formulas (i.e., there is $j, k < i$ such that α_k is of the form $\alpha_j \to \alpha_i$). $\triangleleft$

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- ✓ Sequents: $\Gamma \vdash^{s}_{\mathbf{K}} \varphi$ means that there is a proof of the sequent $\Gamma \Rightarrow \varphi$ where each end point (called a **leaf**) is an axiom.
- ✓ Hilbert systems: $\Gamma \vdash_{\mathbf{K}} \varphi$ means that there is a modal deduction of φ from Γ.

Relationship between proof systems: For all sets of formulas Γ and formulas φ,

$$\Gamma \vdash^{nd}_{\mathbf{K}} \varphi \text{ iff } \Gamma \vdash^{s}_{\mathbf{K}} \varphi \text{ iff } \Gamma \vdash_{\mathbf{K}} \varphi$$

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How do you extend the proof system to other modal logics (e.g., **KT**, **S4**, **S5**)?

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Soundness Theorem: For all Γ and formulas φ, if $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$.

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Completeness Theorem: For all Γ and formulas φ, if $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.
Weak Completeness Theorem: For all formulas φ, if $\models \varphi$ then $\vdash \varphi$.

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Compactness: If $\Gamma \vdash \varphi$, then there is some finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$

Proving Completeness

- Let **K** denote the minimal modal logic and $\vdash \varphi$ mean φ is derivable in **K**. If Γ is a set of formulas, we write $\Gamma \vdash \varphi$ if $\vdash (\psi_1 \wedge \cdots \wedge \psi_k) \to \varphi$ for some finite set $\psi_1, \ldots, \psi_k \in \Gamma$.

- Let Γ be a set of formulas. We write $\Gamma \models \varphi$ provided for all frames $\mathcal{F}$ for all models $\mathcal{M}$ based on $\mathcal{F}$ and all states w in $\mathcal{M}$, $\mathcal{M}, w \models \Gamma$ then $\mathcal{M}, w \models \varphi$.

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- A set of formulas Γ is **consistent** provided $\Gamma \not\vdash \bot$.

- Strong completeness: if $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$ and weak completeness: if $\models \varphi$ then $\vdash \varphi$. Strong completeness implies weak completeness, but weak completeness does not imply strong completeness.

Γ is a **maximally consistent set** if Γ is consistent and for each $\varphi \in \mathcal{L}$ either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$. Alternatively, Γ is consistent and every Γ' such that $\Gamma \subsetneq \Gamma'$ is inconsistent (i.e., every proper superset of Γ is inconsistent).

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Suppose that Γ is a maximally consistent set. Then,

1. If $\vdash \varphi$ then $\varphi \in \Gamma$
2. If $\varphi \to \psi \in \Gamma$ and $\varphi \in \Gamma$ then $\psi \in \Gamma$
3. $\neg\varphi \in \Gamma$ iff $\varphi \notin \Gamma$
4. $\varphi \wedge \psi \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$
5. $\varphi \vee \psi \in \Gamma$ iff $\varphi \in \Gamma$ or $\psi \in \Gamma$