Modal Logic

Eric Pacuit, University of Maryland

October 16, 2023

Modal Proof Theory

✓ Natural deduction: $\Gamma \vdash^{nd}_{\mathbf{K}} \varphi$ means that there is a natural deduction proof where the last line is φ where φ is not in the scope of a subproof and all the assumptions in the proof are from Γ.

2. Sequents
3. Hilbert systems

A **sequent** is two sequence of formulas separated by a double arrow $\Rightarrow$:

$$\varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \psi_k$$

A sequent $\varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \psi_k$ is **valid** when the following formula is valid (true at all states in all models):

$$(\varphi_1 \wedge \cdots \wedge \varphi_n) \rightarrow (\psi_1 \vee \cdots \vee \psi_k)$$

A **sequent** is two sequence of formulas separated by a double arrow $\Rightarrow$:

$$\varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \psi_k$$

A sequent $\varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \psi_k$ is **valid** when the following formula is valid (true at all states in all models):

$$(\varphi_1 \wedge \cdots \wedge \varphi_n) \rightarrow (\psi_1 \vee \cdots \vee \psi_k)$$

$$\frac{\mathcal{A}, \varphi \Rightarrow \mathcal{B}}{\mathcal{A} \Rightarrow \mathcal{B}, \neg\varphi} \qquad \frac{\mathcal{A} \Rightarrow \mathcal{B}, \varphi}{\mathcal{A}, \neg\varphi \Rightarrow \mathcal{B}}$$

$$\frac{\mathcal{A}, \varphi \Rightarrow \mathcal{B}}{\mathcal{A} \Rightarrow \mathcal{B}, \neg\varphi} \qquad \frac{\mathcal{A} \Rightarrow \mathcal{B}, \varphi}{\mathcal{A}, \neg\varphi \Rightarrow \mathcal{B}}$$

$$\frac{\mathcal{A}, \varphi, \psi \Rightarrow \mathcal{B}}{\mathcal{A}, \varphi \wedge \psi \Rightarrow \mathcal{B}} \qquad \frac{\mathcal{A} \Rightarrow \mathcal{B}, \varphi \qquad \mathcal{A} \Rightarrow \mathcal{B}, \psi}{\mathcal{A} \Rightarrow \mathcal{B}, \varphi \wedge \psi}$$

$$\frac{\mathcal{A}, \varphi \Rightarrow \mathcal{B}}{\mathcal{A} \Rightarrow \mathcal{B}, \neg\varphi} \qquad \frac{\mathcal{A} \Rightarrow \mathcal{B}, \varphi}{\mathcal{A}, \neg\varphi \Rightarrow \mathcal{B}}$$

$$\frac{\mathcal{A}, \varphi, \psi \Rightarrow \mathcal{B}}{\mathcal{A}, \varphi \wedge \psi \Rightarrow \mathcal{B}} \qquad \frac{\mathcal{A} \Rightarrow \mathcal{B}, \varphi \qquad \mathcal{A} \Rightarrow \mathcal{B}, \psi}{\mathcal{A} \Rightarrow \mathcal{B}, \varphi \wedge \psi}$$

$$\frac{\mathcal{A} \Rightarrow \mathcal{B}, \varphi, \psi}{\mathcal{A} \Rightarrow \mathcal{B}, \varphi \vee \psi} \qquad \frac{\mathcal{A}, \varphi \Rightarrow \mathcal{B} \qquad \mathcal{A}, \psi \Rightarrow \mathcal{B}}{\mathcal{A}, \varphi \vee \psi \Rightarrow \mathcal{B}}$$

$$\frac{\mathcal{A}, \varphi \Rightarrow \mathcal{B}}{\mathcal{A} \Rightarrow \mathcal{B}, \neg\varphi} \qquad \frac{\mathcal{A} \Rightarrow \mathcal{B}, \varphi}{\mathcal{A}, \neg\varphi \Rightarrow \mathcal{B}}$$

$$\frac{\mathcal{A}, \varphi, \psi \Rightarrow \mathcal{B}}{\mathcal{A}, \varphi \wedge \psi \Rightarrow \mathcal{B}} \qquad \frac{\mathcal{A} \Rightarrow \mathcal{B}, \varphi \qquad \mathcal{A} \Rightarrow \mathcal{B}, \psi}{\mathcal{A} \Rightarrow \mathcal{B}, \varphi \wedge \psi}$$

$$\frac{\mathcal{A} \Rightarrow \mathcal{B}, \varphi, \psi}{\mathcal{A} \Rightarrow \mathcal{B}, \varphi \vee \psi} \qquad \frac{\mathcal{A}, \varphi \Rightarrow \mathcal{B} \qquad \mathcal{A}, \psi \Rightarrow \mathcal{B}}{\mathcal{A}, \varphi \vee \psi \Rightarrow \mathcal{B}}$$

$$\frac{\mathcal{A}, \varphi \Rightarrow \mathcal{B}, \psi}{\mathcal{A} \Rightarrow \mathcal{B}, \varphi \rightarrow \psi} \qquad \frac{\mathcal{A} \Rightarrow \mathcal{B}, \varphi \qquad \mathcal{A}, \psi \Rightarrow \mathcal{B}}{\mathcal{A}, \varphi \rightarrow \psi \Rightarrow \mathcal{B}}$$

$$\frac{\mathcal{A}, \varphi \Rightarrow \mathcal{B}}{\mathcal{A} \Rightarrow \mathcal{B}, \neg\varphi} \qquad \frac{\mathcal{A} \Rightarrow \mathcal{B}, \varphi}{\mathcal{A}, \neg\varphi \Rightarrow \mathcal{B}}$$

$$\frac{\mathcal{A}, \varphi, \psi \Rightarrow \mathcal{B}}{\mathcal{A}, \varphi \wedge \psi \Rightarrow \mathcal{B}} \qquad \frac{\mathcal{A} \Rightarrow \mathcal{B}, \varphi \qquad \mathcal{A} \Rightarrow \mathcal{B}, \psi}{\mathcal{A} \Rightarrow \mathcal{B}, \varphi \wedge \psi}$$

$$\frac{\mathcal{A} \Rightarrow \mathcal{B}, \varphi, \psi}{\mathcal{A} \Rightarrow \mathcal{B}, \varphi \vee \psi} \qquad \frac{\mathcal{A}, \varphi \Rightarrow \mathcal{B} \qquad \mathcal{A}, \psi \Rightarrow \mathcal{B}}{\mathcal{A}, \varphi \vee \psi \Rightarrow \mathcal{B}}$$

$$\frac{\mathcal{A}, \varphi \Rightarrow \mathcal{B}, \psi}{\mathcal{A} \Rightarrow \mathcal{B}, \varphi \to \psi} \qquad \frac{\mathcal{A} \Rightarrow \mathcal{B}, \varphi \qquad \mathcal{A}, \psi \Rightarrow \mathcal{B}}{\mathcal{A}, \varphi \to \psi \Rightarrow \mathcal{B}}$$

There are also *structural rules* that allow us to think of the sequences to the left and right of the $\Rightarrow$ as *sets* of formulas.

$$\frac{\mathcal{A}, \psi, \varphi, \mathcal{C} \Rightarrow \mathcal{B}}{\mathcal{A}, \varphi, \psi, \mathcal{C} \Rightarrow \mathcal{B}} \qquad \frac{\mathcal{A} \Rightarrow \mathcal{B}, \psi, \varphi, \mathcal{C}}{\mathcal{A} \Rightarrow \mathcal{B}, \varphi, \psi, \mathcal{C}}$$

$$\frac{\mathcal{A}, \psi, \varphi, \mathcal{C} \Rightarrow \mathcal{B}}{\mathcal{A}, \varphi, \psi, \mathcal{C} \Rightarrow \mathcal{B}} \qquad \frac{\mathcal{A} \Rightarrow \mathcal{B}, \psi, \varphi, \mathcal{C}}{\mathcal{A} \Rightarrow \mathcal{B}, \varphi, \psi, \mathcal{C}}$$

$$\frac{\mathcal{A}, \mathcal{C} \Rightarrow \mathcal{B}}{\mathcal{A}, \varphi, \mathcal{C} \Rightarrow \mathcal{B}} \qquad \frac{\mathcal{A} \Rightarrow \mathcal{B}, \mathcal{C}}{\mathcal{A} \Rightarrow \mathcal{B}, \varphi, \mathcal{C}}$$

$$\frac{\mathcal{A}, \psi, \varphi, \mathcal{C} \Rightarrow \mathcal{B}}{\mathcal{A}, \varphi, \psi, \mathcal{C} \Rightarrow \mathcal{B}} \qquad \frac{\mathcal{A} \Rightarrow \mathcal{B}, \psi, \varphi, \mathcal{C}}{\mathcal{A} \Rightarrow \mathcal{B}, \varphi, \psi, \mathcal{C}}$$

$$\frac{\mathcal{A}, \mathcal{C} \Rightarrow \mathcal{B}}{\mathcal{A}, \varphi, \mathcal{C} \Rightarrow \mathcal{B}} \qquad \frac{\mathcal{A} \Rightarrow \mathcal{B}, \mathcal{C}}{\mathcal{A} \Rightarrow \mathcal{B}, \varphi, \mathcal{C}}$$

$$\frac{\mathcal{A}, \varphi, \varphi, \mathcal{C} \Rightarrow \mathcal{B}}{\mathcal{A}, \varphi, \mathcal{C} \Rightarrow \mathcal{B}} \qquad \frac{\mathcal{A} \Rightarrow \mathcal{B}, \varphi, \varphi, \mathcal{C}}{\mathcal{A} \Rightarrow \mathcal{B}, \varphi, \mathcal{C}}$$

Key Observation: If a sequent $\mathcal{A} \Rightarrow \mathcal{B}$ consists of propositional formulas, then the previous rules will reduce $\mathcal{A} \Rightarrow \mathcal{B}$ to:

$$p_1, \ldots, p_n \Rightarrow q_1, \ldots, q_k$$

Where each $p_1, \ldots, p_n, q_1, \ldots, q_k$ are all atomic propositions.

Key Observation: If a sequent $\mathcal{A} \Rightarrow \mathcal{B}$ consists of propositional formulas, then the previous rules will reduce $\mathcal{A} \Rightarrow \mathcal{B}$ to:

$$p_1, \ldots, p_n \Rightarrow q_1, \ldots, q_k$$

Where each $p_1, \ldots, p_n, q_1, \ldots, q_k$ are all atomic propositions.

This gives an algorithm to determine whether φ is a tautology.

1. Consider the sequent $\Rightarrow \varphi$.

Key Observation: If a sequent $\mathcal{A} \Rightarrow \mathcal{B}$ consists of propositional formulas, then the previous rules will reduce $\mathcal{A} \Rightarrow \mathcal{B}$ to:

$$p_1, \ldots, p_n \Rightarrow q_1, \ldots, q_k$$

Where each $p_1, \ldots, p_n, q_1, \ldots, q_k$ are all atomic propositions.

This gives an algorithm to determine whether φ is a tautology.

1. Consider the sequent $\Rightarrow \varphi$.
2. Use the previous rules to reduce the sequent to something of the form $\vec{p} \Rightarrow \vec{q}$, where $\vec{p}$ and $\vec{q}$ are both sequences of atomic propositions.

Key Observation: If a sequent $\mathcal{A} \Rightarrow \mathcal{B}$ consists of propositional formulas, then the previous rules will reduce $\mathcal{A} \Rightarrow \mathcal{B}$ to:

$$p_1, \ldots, p_n \Rightarrow q_1, \ldots, q_k$$

Where each $p_1, \ldots, p_n, q_1, \ldots, q_k$ are all atomic propositions.

This gives an algorithm to determine whether φ is a tautology.

1. Consider the sequent $\Rightarrow \varphi$.
2. Use the previous rules to reduce the sequent to something of the form $\vec{p} \Rightarrow \vec{q}$, where $\vec{p}$ and $\vec{q}$ are both sequences of atomic propositions.
3. If there is an atomic proposition that occurs in $\vec{p}$ and $\vec{q}$ then the formula is a tautology. If there is no atomic proposition that occurs in both $\vec{p}$ and $\vec{q}$, then the valuation that makes each atomic proposition in $\vec{p}$ true and each atomic proposition in $\vec{q}$ false is a counterexample.

Modal Sequents

Use the previous rules and the definition of $\Box$ and $\neg\Diamond\neg$ to reduce a sequent to something of the form:

$$\vec{p}, \Diamond\varphi_1, \ldots, \Diamond\varphi_n \Rightarrow \Diamond\psi_1, \ldots, \Diamond\psi_k, \vec{q}$$

Modal Sequents

Use the previous rules and the definition of $\Box$ and $\neg\Diamond\neg$ to reduce a sequent to something of the form:

$$\vec{p}, \Diamond\varphi_1, \ldots, \Diamond\varphi_n \Rightarrow \Diamond\psi_1, \ldots, \Diamond\psi_k, \vec{q}$$

Modal Decomposition Fact. Any sequence of the form

$$\vec{p}, \Diamond\varphi_1, \ldots, \Diamond\varphi_n \Rightarrow \Diamond\psi_1, \ldots, \Diamond\psi_k, \vec{q}$$

is valid if, and only if, either

1. there is an atomic proposition that occurs in both $\vec{p}$ and $\vec{q}$, or
2. for some $1 \leq i \leq n$, $\varphi_i \Rightarrow \psi_1, \ldots, \psi_k$ is valid

Sequent Rule for $\Box$

$$\frac{\mathcal{A} \Rightarrow \varphi}{\Box\mathcal{A} \Rightarrow \Box\varphi}$$

where $\Box\mathcal{A} = \{\Box\psi \mid \psi \in \mathcal{A}\}$

A **proof of the sequent** $\mathcal{A} \Rightarrow \mathcal{B}$ is a tree of sequents where the root is $\mathcal{A} \Rightarrow \mathcal{B}$ and for each sequent in the tree, the children of the sequent are an application of one of the rules.

We call the sequent $\mathcal{A} \Rightarrow \mathcal{B}$ an **axiom** when there is some atomic proposition p that is contained in both $\mathcal{A}$ and $\mathcal{B}$.

Modal Proof Theory

✓ Natural deduction: $\Gamma \vdash^{nd}_{\mathbf{K}} \varphi$ means that there is a natural deduction proof where the last line is φ where φ is not in the scope of a subproof and all the assumptions in the proof are from Γ.

✓ Sequents: $\Gamma \vdash^{s}_{\mathbf{K}} \varphi$ means that there is a proof of the sequent $\Gamma \Rightarrow \varphi$ where each end point (called a **leaf**) is an axiom.

3. Hilbert systems