Modal Logic

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A **logic** is a set of formulas $\mathbf{L}$ satisfying certain closure conditions. We write $\vdash_{\mathbf{L}} \varphi$ iff $\varphi \in \mathbf{L}$.

Rule of inference: "From $\varphi_1, \ldots, \varphi_n$ infer φ", denoted $\dfrac{\varphi_1 \ \varphi_2 \ \cdots \ \varphi_n}{\varphi}$, where $n \geq 0$. A logic is closed under a rule of inference means that if $\{\varphi_1, \varphi_2, \ldots, \varphi_n\} \subseteq \mathbf{L}$, then $\varphi \in \mathbf{L}$

Normal Modal Logic

A **normal modal logic** is a logic that:

- contains all instances of propositional tautologies
- is closed under modus ponens: $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

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- contains all instances of
  - *K*: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
  - *Dual*: $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$
- is closed under necessitation (N): $\dfrac{\varphi}{\Box\varphi}$

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  - *Dual*: $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$
- is closed under necessitation (N): $\dfrac{\varphi}{\Box\varphi}$
- is closed under uniform substitution: $\dfrac{\varphi}{\psi}$, where ψ is obtained from φ by uniformly replacing propositional atoms in φ by arbitrary formulas

Logical consequence

Suppose that Γ is a set of formulas and $\mathbb{F}$ is a set of frames. For a model $\mathcal{M}$ with a state w, we write $\mathcal{M}, w \models \Gamma$ iff $\mathcal{M}, w \models \alpha$ for all $\alpha \in \Gamma$.

Local: $\Gamma \models_{\mathbb{F}} \varphi$ iff for all frames $\mathcal{F} \in \mathbb{F}$, for all models $\mathcal{M}$ based on $\mathcal{F}$ and all states w in $\mathcal{M}$,

$$\mathcal{M}, w \models \Gamma \text{ implies } \mathcal{M}, w \models \varphi$$

Let Γ be a set of formulas. $\mathsf{Fr}(\Gamma) = \{\mathcal{F} \mid \mathcal{F} \models \varphi \text{ for all } \varphi \in \Gamma\}$
For a logic **L**, $\mathsf{Fr}(\mathbf{L})$ is the set of frames that validate all formulas in **L**.

Let Γ be a set of formulas. $\mathsf{Fr}(\Gamma) = \{\mathcal{F} \mid \mathcal{F} \models \varphi \text{ for all } \varphi \in \Gamma\}$
For a logic $\mathbf{L}$, $\mathsf{Fr}(\mathbf{L})$ is the set of frames that validate all formulas in $\mathbf{L}$.

Two Questions:

1. For a logic $\mathbf{L}$, what is the relationship between $\mathbf{L}$ and $\mathsf{Log}(\mathsf{Fr}(\mathbf{L}))$?
2. For a class $\mathbb{F}$ of frames, what is the relationship between $\mathbb{F}$ and $\mathsf{Fr}(\mathsf{Log}(\mathbb{F}))$?

Some Question

- (**Proof theory**) For a normal modal logic **L**, how can we show that $\varphi \in \mathbf{L}$?
- For a normal modal logic **L** and a formula φ, is it *decidable* whether $\varphi \in \mathbf{L}$?

We write $\vdash_{\mathbf{L}} \varphi$ when $\varphi \in \mathbf{L}$.

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For a set of formulas Γ and a logic $\mathbf{L}$, we write $\Gamma \vdash_{\mathbf{L}} \varphi$ when there are $\alpha_1, \ldots, \alpha_n \in \Gamma$ such that $(\alpha_1 \wedge \cdots \wedge \alpha_n) \to \varphi \in \mathbf{L}$.

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For a set of formulas Γ, a formula φ, a set of frames $\mathbb{F}$, and a normal modal logic $\mathbf{L}$, what is the relationship between $\Gamma \vdash_{\mathbf{L}} \varphi$ and $\Gamma \models_{\mathbb{F}} \varphi$?

Modal Proof Theory

1. Natural deduction
2. Sequents
3. Hilbert systems

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For simplicity, focus on the minimal normal modal logic **K**.

Natural Deduction - Rules for Propositional Logic: $\wedge, \rightarrow, \neg$

φ	
ψ	
$\varphi \wedge \psi$	$\wedge$-I

$\varphi \wedge \psi$	
φ	$\wedge$-E

$\varphi \wedge \psi$	
ψ	$\wedge$-E

$\quad \varphi$ (assumption)	
$\quad \psi$	
$\varphi \rightarrow \psi$	$\rightarrow$-I

φ	
$\varphi \rightarrow \psi$	
ψ	$\rightarrow$-E

Natural Deduction - Rules for Propositional Logic: $\neg$

$$
\begin{array}{l|l}
 & \begin{array}{|l} \varphi \\ \hline \bot \end{array} \\
 & \neg\varphi \qquad \neg\text{-I}
\end{array}
\qquad\qquad
\begin{array}{|l}
\varphi \\
\neg\varphi \\
\bot \qquad \neg\text{-E}
\end{array}
$$

Natural Deduction - Rules for Propositional Logic: $\vee$

$\varphi \vee \psi$	
φ	
χ	
ψ	
χ	
χ	$\vee$-E

φ	
$\varphi \vee \psi$	$\vee$-I

ψ	
$\psi \vee \varphi$	$\vee$-I

Natural Deduction - Rules for Propositional Logic: *X*, *IP*, *Reit*

$\bot$

φ *X*

$\neg\varphi$

$\bot$

φ *IP*

φ

φ Reit

φ

φ Reit

1. If I give the order to attack, then, necessarily, there will be a sea battle tomorrow
2. If not, then, necessarily, there will not be one.
3. Now, I give the order or I do not.
4. Hence, either it is necessary that there is a sea battle tomorrow or it is necessary that none occurs.

$$\frac{A \to \Box B \qquad \neg A \to \Box \neg B \qquad A \vee \neg A}{\Box B \vee \Box \neg B}$$

$$\frac{\Box(A \to B) \qquad \Box(\neg A \to \neg B) \qquad A \vee \neg A}{\Box B \vee \Box \neg B}$$

Which argument is valid?

1	$A \to \Box B$	Assumption
2	$\neg A \to \Box \neg B$	Assumption
3		

1	$A \to \Box B$	Assumption
2	$\neg A \to \Box \neg B$	Assumption
3	$A \vee \neg A$	Assumption
4		

1	$A \to \Box B$	Assumption
2	$\neg A \to \Box \neg B$	Assumption
3	$A \lor \neg A$	Assumption
4	$\quad A$	
5		

1	$A \to \Box B$	Assumption
2	$\neg A \to \Box\neg B$	Assumption
3	$A \vee \neg A$	Assumption
4	$\quad A$	
5	$\quad A \to \Box B$	Reit: 1
6		

1	$A \to \Box B$	Assumption
2	$\neg A \to \Box \neg B$	Assumption
3	$A \vee \neg A$	Assumption
4	$\quad A$	
5	$\quad A \to \Box B$	Reit: 1
6	$\quad \Box B$	$\to$-E: 4,5
7		

1	$A \to \Box B$	Assumption
2	$\neg A \to \Box \neg B$	Assumption
3	$A \vee \neg A$	Assumption
4	$\quad A$	
5	$\quad A \to \Box B$	Reit: 1
6	$\quad \Box B$	$\to$-E: 4,5
7	$\quad \Box B \vee \Box \neg B$	$\vee$-I: 6
8		

1	$A \to \Box B$	Assumption
2	$\neg A \to \Box\neg B$	Assumption
3	$A \vee \neg A$	Assumption
4	$\quad A$	
5	$\quad A \to \Box B$	Reit: 1
6	$\quad \Box B$	$\to$-E: 4,5
7	$\quad \Box B \vee \Box\neg B$	$\vee$-I: 6
8	$\quad \neg A$	
9		

1	$A \to \Box B$	Assumption
2	$\neg A \to \Box\neg B$	Assumption
3	$A \vee \neg A$	Assumption
4	$\quad A$	
5	$\quad A \to \Box B$	Reit: 1
6	$\quad \Box B$	$\to$-E: 4,5
7	$\quad \Box B \vee \Box\neg B$	$\vee$-I: 6
8	$\quad \neg A$	
9	$\quad \neg A \to \Box\neg B$	Reit: 2
10		

1	$A \rightarrow \Box B$	Assumption
2	$\neg A \rightarrow \Box\neg B$	Assumption
3	$A \vee \neg A$	Assumption
4	$\quad A$	
5	$\quad A \rightarrow \Box B$	Reit: 1
6	$\quad \Box B$	$\rightarrow$-E: 4,5
7	$\quad \Box B \vee \Box\neg B$	$\vee$-I: 6
8	$\quad \neg A$	
9	$\quad \neg A \rightarrow \Box\neg B$	Reit: 2
10	$\quad \Box\neg B$	$\vee$-I: 9
11		

1	$A \to \Box B$	Assumption
2	$\neg A \to \Box \neg B$	Assumption
3	$A \lor \neg A$	Assumption
4	A	
5	$A \to \Box B$	Reit: 1
6	$\Box B$	$\to$-E: 4,5
7	$\Box B \lor \Box \neg B$	$\lor$-I: 6
8	$\neg A$	
9	$\neg A \to \Box \neg B$	Reit: 2
10	$\Box \neg B$	$\lor$-I: 9
11	$\Box B \lor \Box \neg B$	$\lor$-I: 10
12		

1	$A \to \Box B$	Assumption
2	$\neg A \to \Box\neg B$	Assumption
3	$A \vee \neg A$	Assumption
4	A	
5	$A \to \Box B$	Reit: 1
6	$\Box B$	$\to$-E: 4,5
7	$\Box B \vee \Box\neg B$	$\vee$-I: 6
8	$\neg A$	
9	$\neg A \to \Box\neg B$	Reit: 2
10	$\Box\neg B$	$\vee$-I: 9
11	$\Box B \vee \Box\neg B$	$\vee$-I: 10
12	$\Box B \vee \Box\neg B$	$\vee$-E: 3,7,11

1 $\Box(A \to B)$

2 $\Box(\neg A \to \neg B)$

3

1 $\Box(A \to B)$

2 $\Box(\neg A \to \neg B)$

3 $A \vee \neg A$

4

1	$\Box(A \to B)$
2	$\Box(\neg A \to \neg B)$
3	$A \lor \neg A$
4	$\quad A$
5	

1 $\Box(A \to B)$

2 $\Box(\neg A \to \neg B)$

3 $A \vee \neg A$

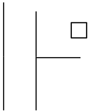
4 $\quad A$

5 $\quad \Box(A \to B)$

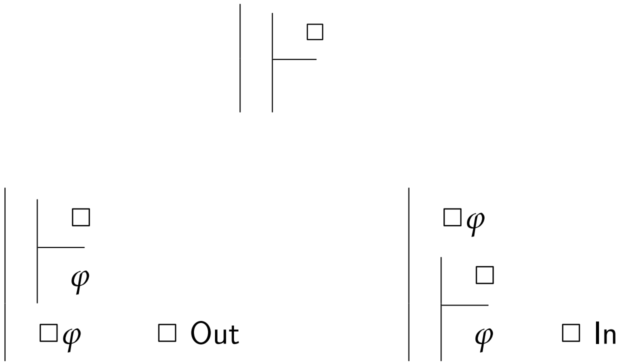
1		$\Box(A \to B)$
2		$\Box(\neg A \to \neg B)$
3		$A \vee \neg A$
4		A
5		$\Box(A \to B)$

What do we do now?

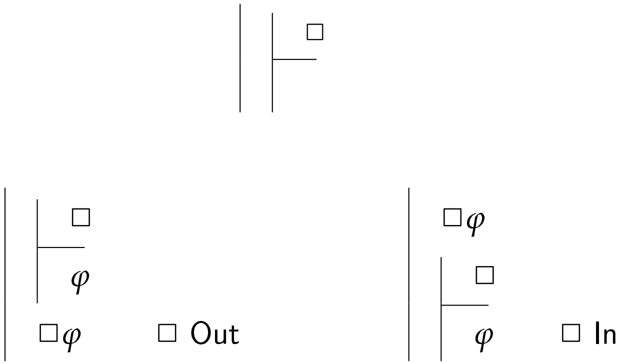
New idea: boxed subproof



New idea: boxed subproof



New idea: boxed subproof



Plus: modify the Reiteration rule so that you cannot reiterate a formula in a boxed-subproof.

Be Careful

1	p
2	

Be Careful

1	p
2	$\square$
3	

Be Careful

1	p	
2	$\quad \square$	
3	$\quad p$	Reit: 1
4		

Be Careful

1	p	
2	$\quad \Box$	
3	$\quad p$	Reit: 1
4	$\Box p$	$\Box$ Out: 3

Cannot move non-boxed formulas into boxed subproofs!

Be Careful

1	p	
2	$\quad \Box$	
3	$\quad p$	Reit: 1
4	$\Box p$	$\Box$ Out: 3

Cannot move non-boxed formulas into boxed subproofs!

But you can, *if there is a derivation of* p!

Be Careful

Suppose $p \to q$ is derivable ($\vdash p \to q$), then so is $\Box p \to \Box q$.

$$\frac{\vdash^{nd}_{\mathbf{K}} \varphi \to \psi}{\vdash^{nd}_{\mathbf{K}} \Box\varphi \to \Box\psi}$$

Be Careful

Suppose $p \to q$ is derivable ($\vdash p \to q$), then so is $\Box p \to \Box q$.

$$\frac{\vdash^{nd}_{\mathbf{K}} \varphi \to \psi}{\vdash^{nd}_{\mathbf{K}} \Box\varphi \to \Box\psi}$$

1	$\Box\varphi$	
2	$\Box$	
3	$\varphi \to \psi$	Insert proof of $\varphi \to \psi$
4	φ	$\Box$ In: 1
5	ψ	$\to$-E: 3,4
6	$\Box\psi$	$\Box$ Out: 5
7	$\Box\varphi \to \Box\psi$	$\to$-I: 1-6

The Second Version of the Sea Battle Argument

1	$\Box(A \to B)$
2	$\Box(\neg A \to \neg B)$
3	$A \vee \neg A$
4	$\quad A$
5	

The Second Version of the Sea Battle Argument

1	$\Box(A \to B)$
2	$\Box(\neg A \to \neg B)$
3	$A \lor \neg A$
4	$\quad A$
5	$\quad \Box(A \to B)$
6	$\quad\quad \Box$
7	$\quad\quad A \to B$

$$\vdash^{nd}_{\mathbf{K}} (\Box p \wedge \Box q) \to \Box(p \wedge q)$$

1	$\Box p \wedge \Box q$	Assumption
2	$\Box p$	$\wedge$-E: 1
3		

$$\vdash^{nd}_{\mathbf{K}} (\Box p \wedge \Box q) \to \Box(p \wedge q)$$

1	$\Box p \wedge \Box q$	Assumption
2	$\Box p$	$\wedge$-E: 1
3	$\Box q$	$\wedge$-E: 1
4		

$$\vdash^{nd}_{\mathbf{K}} (\Box p \wedge \Box q) \rightarrow \Box(p \wedge q)$$

1	$\Box p \wedge \Box q$	Assumption
2	$\Box p$	$\wedge$-E: 1
3	$\Box q$	$\wedge$-E: 1
4	$\Box$	
5		

$$\vdash^{nd}_{\mathbf{K}} (\Box p \land \Box q) \to \Box(p \land q)$$

1	$\Box p \land \Box q$	Assumption
2	$\Box p$	$\land$-E: 1
3	$\Box q$	$\land$-E: 1
4	$\quad \Box$	
5	$\quad p$	$\Box$ In: 2
6		

$$\vdash^{nd}_{\mathbf{K}} (\Box p \wedge \Box q) \rightarrow \Box(p \wedge q)$$

1	$\Box p \wedge \Box q$	Assumption
2	$\Box p$	$\wedge$-E: 1
3	$\Box q$	$\wedge$-E: 1
4	$\Box$	
5	p	$\Box$ In: 2
6	q	$\Box$ In: 3
7		

$$\vdash^{nd}_{\mathbf{K}} (\Box p \wedge \Box q) \rightarrow \Box(p \wedge q)$$

1	$\quad \Box p \wedge \Box q$	Assumption
2	$\quad \Box p$	$\wedge$-E: 1
3	$\quad \Box q$	$\wedge$-E: 1
4	$\quad\quad \Box$	
5	$\quad\quad p$	$\Box$ In: 2
6	$\quad\quad q$	$\Box$ In: 3
7	$\quad\quad p \wedge q$	$\wedge$-I: 5,6
8	$\quad \Box(p \wedge q)$	$\Box$ Out: 7
9	$\Box p \wedge \Box q \rightarrow \Box(p \wedge q)$	$\rightarrow$-I: 1-8

1	$\Box(p \wedge q)$	Assumption
2	$\Box$	
3		

$$\vdash^{nd}_{\mathbf{K}} \Box(p \wedge q) \rightarrow (\Box p \wedge \Box q)$$

1	$\quad\quad \Box(p \wedge q)$	Assumption
2	$\quad\quad\quad \Box$	
3	$\quad\quad\quad p \wedge q$	$\Box$ In: 1
4		

$$\vdash^{nd}_{\mathbf{K}} \Box(p \wedge q) \rightarrow (\Box p \wedge \Box q)$$

1	$\Box(p \wedge q)$	Assumption
2	$\Box$	
3	$p \wedge q$	$\Box$ In: 1
4	p	$\wedge$-E: 3
5		

$$\vdash^{nd}_{\mathbf{K}} \Box(p \wedge q) \rightarrow (\Box p \wedge \Box q)$$

1	$\Box(p \wedge q)$	Assumption
2	$\Box$	
3	$p \wedge q$	$\Box$ In: 1
4	p	$\wedge$-E: 3
5	$\Box p$	$\Box$ Out: 4
6		

$$\vdash^{nd}_{\mathbf{K}} \Box(p \wedge q) \rightarrow (\Box p \wedge \Box q)$$

$\vdash^{nd}_{\mathbf{K}} \Box(p \wedge q) \to (\Box p \wedge \Box q)$

1	$\Box(p \wedge q)$	Assumption
2	$\Box$	
3	$p \wedge q$	$\Box$ In: 1
4	p	$\wedge$-E: 3
5	$\Box p$	$\Box$ Out: 4
6	$\Box$	
7		

$\vdash^{nd}_{\mathbf{K}} \Box(p \wedge q) \rightarrow (\Box p \wedge \Box q)$

1	$\Box(p \wedge q)$	Assumption
2	$\Box$	
3	$p \wedge q$	$\Box$ In: 1
4	p	$\wedge$-E: 3
5	$\Box p$	$\Box$ Out: 4
6	$\Box$	
7	$p \wedge q$	$\Box$ In: 1
8		

$\vdash^{nd}_{\mathbf{K}} \Box(p \wedge q) \rightarrow (\Box p \wedge \Box q)$

1	$\Box(p \wedge q)$	Assumption
2	$\quad \Box$	
3	$\quad p \wedge q$	$\Box$ In: 1
4	$\quad p$	$\wedge$-E: 3
5	$\Box p$	$\Box$ Out: 4
6	$\quad \Box$	
7	$\quad p \wedge q$	$\Box$ In: 1
8	$\quad q$	$\wedge$-E: 7
9		

$$\vdash^{nd}_{\mathbf{K}} \Box(p \land q) \to (\Box p \land \Box q)$$

1	$\Box(p \land q)$	Assumption
2	$\Box$	
3	$p \land q$	$\Box$ In: 1
4	p	$\land$-E: 3
5	$\Box p$	$\Box$ Out: 4
6	$\Box$	
7	$p \land q$	$\Box$ In: 1
8	q	$\land$-E: 7
9	$\Box q$	$\Box$ Out: 8
10		

$$\vdash^{nd}_{\mathbf{K}} \Box(p \wedge q) \to (\Box p \wedge \Box q)$$

1	$\mid\ \Box(p \wedge q)$	Assumption
2	$\mid\ \mid\ \Box$	
3	$\mid\ \mid\ p \wedge q$	$\Box$ In: 1
4	$\mid\ \mid\ p$	$\wedge$-E: 3
5	$\mid\ \Box p$	$\Box$ Out: 4
6	$\mid\ \mid\ \Box$	
7	$\mid\ \mid\ p \wedge q$	$\Box$ In: 1
8	$\mid\ \mid\ q$	$\wedge$-E: 7
9	$\mid\ \Box q$	$\Box$ Out: 8
10	$\mid\ \Box p \wedge \Box q$	$\wedge$-I: 4,8
11	$\Box(p \wedge q) \to (\Box p \wedge \Box q)$	$\to$-I: 1-10

$$\vdash^{sd}_{\mathbf{K}} \Box(p \to q) \to (\Box p \to \Box q)$$

1	$\Box(p \to q)$
2	$\Box p$
3	

$$\vdash^{sd}_{\mathbf{K}} \Box(p \to q) \to (\Box p \to \Box q)$$

1	$\Box(p \to q)$
2	$\Box p$
3	$\Box$
4	

$$\vdash^{sd}_{\mathbf{K}} \Box(p \to q) \to (\Box p \to \Box q)$$

1	$\Box(p \to q)$	
2	$\Box p$	
3	$\Box$	
4	$p \to q$	$\Box$ In: 1
5		

$$\vdash^{sd}_{\mathbf{K}} \Box(p \to q) \to (\Box p \to \Box q)$$

1	$\Box(p \to q)$	
2	$\Box p$	
3	$\Box$	
4	$p \to q$	$\Box$ In: 1
5	p	$\Box$ In: 2
6		

$$\vdash^{sd}_{\mathbf{K}} \Box(p \to q) \to (\Box p \to \Box q)$$

1	$\Box(p \to q)$	
2	$\Box p$	
3	$\Box$	
4	$p \to q$	$\Box$ In: 1
5	p	$\Box$ In: 2
6	q	$\to$-E: 4,5
7	$\Box q$	$\Box$ Out: 6
8	$\Box p \to \Box q$	$\to$-I: 2-7
9	$\Box(p \to q) \to (\Box p \to \Box q)$	$\to$-I: 1-8

Modal Proof Theory

✓ Natural deduction: $\Gamma \vdash^{nd}_{\mathbf{K}} \varphi$ means that there is a natural deduction proof where the last line is φ where φ is not in the scope of a subproof and all the assumptions in the proof are from Γ.

2. Sequents
3. Hilbert systems

A **sequent** is two sequence of formulas separated by a double arrow $\Rightarrow$:

$$\varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \psi_k$$

A sequent $\varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \psi_k$ is **valid** when the following formula is valid (true at all states in all models):

$$(\varphi_1 \wedge \cdots \wedge \varphi_n) \rightarrow (\psi_1 \vee \cdots \vee \psi_k)$$

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A sequent $\varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \psi_k$ is **valid** when the following formula is valid (true at all states in all models):

$$(\varphi_1 \wedge \cdots \wedge \varphi_n) \to (\psi_1 \vee \cdots \vee \psi_k)$$

$$\frac{\mathcal{A}, \varphi \Rightarrow \mathcal{B}}{\mathcal{A} \Rightarrow \mathcal{B}, \neg\varphi} \qquad \frac{\mathcal{A} \Rightarrow \mathcal{B}, \varphi}{\mathcal{A}, \neg\varphi \Rightarrow \mathcal{B}}$$

$$\frac{\mathcal{A}, \varphi \Rightarrow \mathcal{B}}{\mathcal{A} \Rightarrow \mathcal{B}, \neg\varphi} \qquad \frac{\mathcal{A} \Rightarrow \mathcal{B}, \varphi}{\mathcal{A}, \neg\varphi \Rightarrow \mathcal{B}}$$

$$\frac{\mathcal{A}, \varphi, \psi \Rightarrow \mathcal{B}}{\mathcal{A}, \varphi \wedge \psi \Rightarrow \mathcal{B}} \qquad \frac{\mathcal{A} \Rightarrow \mathcal{B}, \varphi \qquad \mathcal{A} \Rightarrow \mathcal{B}, \psi}{\mathcal{A} \Rightarrow \mathcal{B}, \varphi \wedge \psi}$$

$$\frac{\mathcal{A}, \varphi \Rightarrow \mathcal{B}}{\mathcal{A} \Rightarrow \mathcal{B}, \neg\varphi} \qquad \frac{\mathcal{A} \Rightarrow \mathcal{B}, \varphi}{\mathcal{A}, \neg\varphi \Rightarrow \mathcal{B}}$$

$$\frac{\mathcal{A}, \varphi, \psi \Rightarrow \mathcal{B}}{\mathcal{A}, \varphi \wedge \psi \Rightarrow \mathcal{B}} \qquad \frac{\mathcal{A} \Rightarrow \mathcal{B}, \varphi \qquad \mathcal{A} \Rightarrow \mathcal{B}, \psi}{\mathcal{A} \Rightarrow \mathcal{B}, \varphi \wedge \psi}$$

$$\frac{\mathcal{A} \Rightarrow \mathcal{B}, \varphi, \psi}{\mathcal{A} \Rightarrow \mathcal{B}, \varphi \vee \psi} \qquad \frac{\mathcal{A}, \varphi \Rightarrow \mathcal{B} \qquad \mathcal{A}, \psi \Rightarrow \mathcal{B}}{\mathcal{A}, \varphi \vee \psi \Rightarrow \mathcal{B}}$$

$$\frac{\mathcal{A}, \varphi \Rightarrow \mathcal{B}}{\mathcal{A} \Rightarrow \mathcal{B}, \neg\varphi} \qquad \frac{\mathcal{A} \Rightarrow \mathcal{B}, \varphi}{\mathcal{A}, \neg\varphi \Rightarrow \mathcal{B}}$$

$$\frac{\mathcal{A}, \varphi, \psi \Rightarrow \mathcal{B}}{\mathcal{A}, \varphi \wedge \psi \Rightarrow \mathcal{B}} \qquad \frac{\mathcal{A} \Rightarrow \mathcal{B}, \varphi \qquad \mathcal{A} \Rightarrow \mathcal{B}, \psi}{\mathcal{A} \Rightarrow \mathcal{B}, \varphi \wedge \psi}$$

$$\frac{\mathcal{A} \Rightarrow \mathcal{B}, \varphi, \psi}{\mathcal{A} \Rightarrow \mathcal{B}, \varphi \vee \psi} \qquad \frac{\mathcal{A}, \varphi \Rightarrow \mathcal{B} \qquad \mathcal{A}, \psi \Rightarrow \mathcal{B}}{\mathcal{A}, \varphi \vee \psi \Rightarrow \mathcal{B}}$$

$$\frac{\mathcal{A}, \varphi \Rightarrow \mathcal{B}, \psi}{\mathcal{A} \Rightarrow \mathcal{B}, \varphi \rightarrow \psi} \qquad \frac{\mathcal{A} \Rightarrow \mathcal{B}, \varphi \qquad \mathcal{A}, \psi \Rightarrow \mathcal{B}}{\mathcal{A}, \varphi \rightarrow \psi \Rightarrow \mathcal{B}}$$

$$\frac{\mathcal{A}, \varphi \Rightarrow \mathcal{B}}{\mathcal{A} \Rightarrow \mathcal{B}, \neg\varphi} \qquad \frac{\mathcal{A} \Rightarrow \mathcal{B}, \varphi}{\mathcal{A}, \neg\varphi \Rightarrow \mathcal{B}}$$

$$\frac{\mathcal{A}, \varphi, \psi \Rightarrow \mathcal{B}}{\mathcal{A}, \varphi \wedge \psi \Rightarrow \mathcal{B}} \qquad \frac{\mathcal{A} \Rightarrow \mathcal{B}, \varphi \qquad \mathcal{A} \Rightarrow \mathcal{B}, \psi}{\mathcal{A} \Rightarrow \mathcal{B}, \varphi \wedge \psi}$$

$$\frac{\mathcal{A} \Rightarrow \mathcal{B}, \varphi, \psi}{\mathcal{A} \Rightarrow \mathcal{B}, \varphi \vee \psi} \qquad \frac{\mathcal{A}, \varphi \Rightarrow \mathcal{B} \qquad \mathcal{A}, \psi \Rightarrow \mathcal{B}}{\mathcal{A}, \varphi \vee \psi \Rightarrow \mathcal{B}}$$

$$\frac{\mathcal{A}, \varphi \Rightarrow \mathcal{B}, \psi}{\mathcal{A} \Rightarrow \mathcal{B}, \varphi \rightarrow \psi} \qquad \frac{\mathcal{A} \Rightarrow \mathcal{B}, \varphi \qquad \mathcal{A}, \psi \Rightarrow \mathcal{B}}{\mathcal{A}, \varphi \rightarrow \psi \Rightarrow \mathcal{B}}$$

There are also *structural rules* that allow us to think of the sequences to the left and right of the $\Rightarrow$ as *sets* of formulas.