Modal Logic

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Logics: Soundness and Completeness

A **logic** is a set of formulas **L** satisfying certain closure conditions. We write $\vdash_{\mathbf{L}} \varphi$ iff $\varphi \in \mathbf{L}$.

Rule of inference: "From $\varphi_1, \dots, \varphi_n$ infer φ", denoted $\dfrac{\varphi_1 \ \varphi_2 \ \cdots \ \varphi_n}{\varphi}$, where $n \geq 0$. A logic is closed under a rule of inference means that if $\{\varphi_1, \varphi_2, \dots, \varphi_n\} \subseteq \mathbf{L}$, then $\varphi \in \mathbf{L}$

Uniform Substitution (US)

$$\frac{\varphi}{\psi}$$

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Axiom Schemes vs. Axioms:

- The logic contains all instances of $\alpha \to (\beta \to \alpha)$
- The logic contains the axiom $p \to (q \to p)$ and is closed under uniform substitution

Propositional Calculus (PC)

A modal formula φ is called a **(propositional) tautology** if $\varphi = (\alpha)^\sigma$ where σ is a substitution, α is a formula of propositional logic and α is a tautology.

For example, $\Box p \to (\Diamond(p \land q) \to \Box p)$ is a tautology because $a \to (b \to a)$ is a tautology in the language of propositional logic and

$$(a \to (b \to a))^\sigma = \Box p \to (\Diamond(p \land q) \to \Box p)$$

where $\sigma(a) = \Box p$ and $\sigma(b) = \Diamond(p \land q)$.

Propositional Calculus (PC)

RPL $\quad \dfrac{\varphi_1 \quad \varphi_2 \quad \cdots \quad \varphi_n}{\varphi}$, where φ is a tautological consequence of $\varphi_1, \ldots, \varphi_n$ (i.e., $(\varphi_1 \wedge \cdots \wedge \varphi_n) \rightarrow \varphi$ is a propositional tautology).

Propositional Calculus (PC)

→1. $\alpha \to (\beta \to \alpha)$

→2. $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$

→3. $\bot \to \alpha$

∧1. $(\alpha \wedge \psi) \to \alpha$

∧2. $(\alpha \wedge \beta) \to \beta$

∧3. $\alpha \to (\beta \to (\alpha \wedge \beta))$

∨1. $\alpha \to (\alpha \vee \beta)$

∨2. $\beta \to (\alpha \vee \beta)$

∨3. $(\alpha \to \bot) \to ((\beta \to \bot) \to ((\alpha \vee \beta) \to \bot)$

DN. $((\alpha \to \bot) \to \bot) \to \alpha$

MP. (Modus Ponens) $\dfrac{\alpha \qquad \alpha \to \beta}{\beta}$

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- is closed under necessitation (N): $\dfrac{\varphi}{\Box\varphi}$
- is closed under uniform substitution: $\dfrac{\varphi}{\psi}$, where ψ is obtained from φ by uniformly replacing propositional atoms in φ by arbitrary formulas

An equivalent definition of a normal modal logic: A **normal modal logic** is a logic that:

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  - *Dual*: $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$
- is closed under *RK*: $\dfrac{(\varphi_1 \wedge \cdots \wedge \varphi_n) \to \varphi}{(\Box\varphi_1 \wedge \cdots \wedge \Box\varphi_n) \to \Box\varphi}$ $(n \geq 0)$

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  - *Dual*: $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$,
  - *M*: $\Box(\varphi \land \psi) \to (\Box\varphi \land \Box\psi)$
  - *C*: $(\Box\varphi \land \Box\psi) \to \Box(\varphi \land \psi)$
  - *N*: $\Box\top$
- is closed under *RE*: $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

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4. Let **K** be the smallest normal modal logic: The smallest set of formulas containing all propositional tautologies, all instances of *K*, all instances of *Dual*, closed under Modus Ponens, and closed under Necessitation.

Modal Logics

PC: All propositional tautologies

N: The rule of necessitation: $\dfrac{\varphi}{\Box\varphi}$

Some Axioms

K	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
D	$\Box\varphi \to \Diamond\varphi$
T	$\Box\varphi \to \varphi$
4	$\Box\varphi \to \Box\Box\varphi$
5	$\neg\Box\varphi \to \Box\neg\Box\varphi$
L	$\Box(\Box\varphi \to \varphi) \to \Box\varphi$

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Some Normal Modal Logics

K	$K + PC + N$
T	$K + T + PC + N$
K4	$K + 4 + PC + N$
S4	$K + T + 4 + PC + N$
S5	$K + T + 4 + 5 + PC + N$
KD45	$K + D + 4 + 5 + PC + N$
GL	$K + L + PC + N$

Suppose that $\mathbf{L}$ and $\mathbf{L}'$ are two modal logics. We say that $\mathbf{L}'$ **extends** $\mathbf{L}$ when $\mathbf{L} \subseteq \mathbf{L}'$. For example, Suppose that $\mathbf{L}$ and $\mathbf{L}'$ are two modal logics. We say that $\mathbf{L}'$ **extends** $\mathbf{L}$ when $\mathbf{L} \subseteq \mathbf{L}'$. For example,

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- $\mathbf{K} \subseteq \mathbf{T} \subseteq \mathbf{S4} \subseteq \mathbf{S5}$
- Let $\mathbb{F}_{ref} = \{\mathcal{F} \mid \mathcal{F} \text{ is reflexive}\}$. Then, $\mathbf{T} \subseteq \mathrm{Log}(\mathbb{F}_{ref})$
- Let $\mathbb{F}_{ref,trans} = \{\mathcal{F} \mid \mathcal{F} \text{ is reflexive and transitive}\}$. Then, $\mathbf{S4} \subseteq \mathrm{Log}(\mathbb{F}_{ref,trans})$
- Let $\mathbb{F}_{equiv} = \{\mathcal{F} \mid \mathcal{F} \text{ is an equivalence relation}\}$. Then, $\mathbf{S5} \subseteq \mathrm{Log}(\mathbb{F}_{equiv})$

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What is the relationship between **GL** and **K4**? **K4** $\subseteq$ **GL**

$\Box\varphi \to \Box\Box\varphi$ is a *consequence* of $\Box(\Box\varphi \to \varphi) \to \Box\varphi$

Logical consequence

Suppose that Γ is a set of formulas and $\mathbb{F}$ is a set of frames. We write $\mathcal{M}, w \models \Gamma$ iff $\mathcal{M}, w \models \alpha$ for all $\alpha \in \Gamma$.

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Local: $\Gamma \models_{\mathbb{F}} \varphi$ iff for all frames $\mathcal{F} \in \mathbb{F}$, for all models $\mathcal{M}$ based on $\mathcal{F}$ and all states w in $\mathcal{M}$,

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Global: $\Gamma \models^{g}_{\mathbb{F}} \varphi$ iff for all frames $\mathcal{F} \in \mathbb{F}$, for all models $\mathcal{M}$ based on $\mathcal{F}$,

$$\mathcal{M} \models \Gamma \text{ implies } \mathcal{M} \models \varphi$$

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Examples

- $\models (\Box p \wedge \Diamond q) \rightarrow \Diamond(p \wedge q)$
- $\{\Box p \rightarrow \Diamond p\} \models \Diamond\top$
- $\{\Box p \rightarrow p\} \models \Box p \rightarrow \Diamond p$

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- $\{\Box p \to \Diamond p, \Diamond p \to \Box\Diamond p\} \models \Box(\Box p \to p)$

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- $\{\Box p \to \Diamond p, \Diamond p \to \Box\Diamond p\} \models \Box(\Box p \to p)$
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