Modal Logic

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Standard Translation

$st_x : \mathcal{L} \to \mathcal{L}_1$

First-order language with an appropriate signature

$$
\begin{aligned}
st_x(p) &= Px \\
st_x(\neg\varphi) &= \neg st_x(\varphi) \\
st_x(\varphi \wedge \psi) &= st_x(\varphi) \wedge st_x(\psi) \\
st_x(\Box\varphi) &= \forall y(xRy \to st_y(\varphi)) \\
st_x(\Diamond\varphi) &= \exists y(xRy \wedge st_y(\varphi))
\end{aligned}
$$

$$
\begin{aligned}
st_y(p) &= Py \\
st_y(\neg\varphi) &= \neg st_y(\varphi) \\
st_y(\varphi \wedge \psi) &= st_y(\varphi) \wedge st_x(\psi) \\
st_y(\Box\varphi) &= \forall x(yRx \to st_x(\varphi)) \\
st_y(\Diamond\varphi) &= \exists x(yRx \wedge st_x(\varphi))
\end{aligned}
$$

Fact. Modal logic falls in the two-variable fragment of first-order logic.

Lemma. For each $w \in W$, $\mathcal{M}, w \models \varphi$ iff $\mathcal{M} \Vdash st_x(\varphi)[x/w]$.

Lemma. $\mathcal{F} \models \varphi$ iff $\mathcal{F} \Vdash \forall P_1 \forall P_2 \cdots \forall P_n \forall x\ st_x(\varphi)$
where the P_i correspond to the atomic propositions p_i in φ, let $ST(\varphi)$ be the Second-Order translation of φ. L

$$ST(p \to \Diamond p) \quad = \quad \forall P \forall x \; st_x(p \to \Diamond p)$$

$$
\begin{aligned}
ST(p \to \Diamond p) &= \forall P \forall x \; st_x(p \to \Diamond p) \\
&= \forall P \forall x \; (st_x(p) \to st_x(\Diamond p))
\end{aligned}
$$

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ST(p \to \Diamond p) &= \forall P \forall x \; st_x(p \to \Diamond p) \\
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&= \forall P \forall x \; (st_x(p) \rightarrow st_x(\Diamond p)) \\
&= \forall P \forall x \; (Px \rightarrow st_x(\Diamond p)) \\
&= \forall P \forall x \; (Px \rightarrow \exists y(x \, R \, y \wedge st_y(p)))
\end{aligned}
$$

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ST(p \to \Diamond p) &= \forall P \forall x \; st_x(p \to \Diamond p) \\
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&= \forall P \forall x \; (Px \to \exists y(x \; R \; y \wedge st_y(p))) \\
&= \forall P \forall x \; (Px \to \exists y(x \; R \; y \wedge Py))
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\end{aligned}
$$

$$
\mathcal{F} \Vdash \forall x \; x \, R \, x \text{ iff } \mathcal{F} \models p \to \Diamond p \text{ iff } \mathcal{F} \Vdash \forall P \forall x \; (Px \to \exists y(x \, R \, y \wedge Py))
$$

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ST(p \to \Diamond p) &= \forall P \forall x \; st_x(p \to \Diamond p) \\
&= \forall P \forall x \; (st_x(p) \to st_x(\Diamond p)) \\
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\end{aligned}
$$

$$\mathcal{F} \Vdash \forall x \; x \; R \; x \text{ iff } \mathcal{F} \models p \to \Diamond p \text{ iff } \mathcal{F} \Vdash \forall P \forall x \; (Px \to \exists y(x \; R \; y \wedge Py))$$

So, reflexivity is *equivalent* to $\forall P \forall x \; (Px \to \exists y(x \; R \; y \wedge Py))$.