Introduction to Modal Logic

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Valid in a class F of frames:

$\models_{\mathsf{F}} \varphi$: for all $\mathcal{F} \in \mathsf{F}$, $\mathcal{F} \models \varphi$

Note that if $\mathcal{F} \models \varphi$ where φ is some modal formula, then $\mathcal{F} \models \varphi^*$ where φ^* is any **substitution instance** of φ. That is, φ^* is obtained by replacing sentence letters in φ with modal formulas.

A **substitution** is a function from sentence letters to well formed modal formulas (i.e., $\sigma : \mathsf{At} \to \mathcal{L}$). We extend a substitution σ to all formulas φ by recursion as follows (we write φ^σ for $\sigma(\varphi)$):

1. $\sigma(\bot) = \bot$
2. $\sigma(\neg\varphi) = \neg\sigma(\varphi)$
3. $\sigma(\varphi \wedge \psi) = \sigma(\varphi) \wedge \sigma(\psi)$
4. $\sigma(\Box\varphi) = \Box\sigma(\varphi)$
5. $\sigma(\Diamond\varphi) = \Diamond\sigma(\varphi)$ $\triangleleft$

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For example, if $\sigma(p) = \Box\Diamond(p \wedge q)$ and $\sigma(q) = p \wedge \Box q$ then

$$(\Box(p \wedge q) \to \Box p)^\sigma = \Box((\Box\Diamond(p \wedge q)) \wedge (p \wedge \Box q)) \to \Box(\Box\Diamond(p \wedge q))$$

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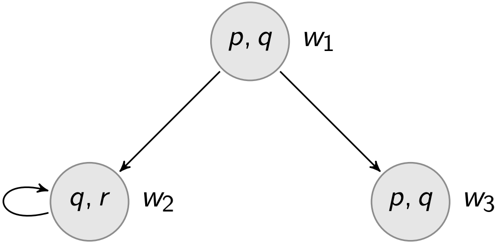
This means, for instance, that in order to show that $\mathcal{F} \not\models \Box\varphi \to \varphi$ it is enough to show that $\mathcal{F} \not\models \Box p \to p$ where p is a sentence letter.

Model validity



$$\mathcal{M} \models \Box q$$

validity on a model is *not* closed under substitution ($\mathcal{M} \not\models \Box p$)

Frame validity

Some frame validities:

- $\Box\top$
- $\Box p \leftrightarrow \neg\Diamond\neg p$
- $(\Box p \wedge \Box q) \leftrightarrow \Box(p \wedge q)$
- $\Box(p \to q) \to (\Box p \to \Box q)$

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Some frame non-validities:

- $\Box p \vee \Box\neg p$ (compare with the validity $\Box p \vee \neg\Box p$)
- $(\Diamond p \wedge \Diamond q) \rightarrow \Diamond(p \wedge q)$
- $\Box(p \vee q) \rightarrow (\Box p \vee \Box q)$
- $\Box p \rightarrow p$
- $\Box p \rightarrow \Diamond p$

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Correspondence

Fact. For any frame $\mathcal{F} = \langle W, R \rangle$,

$$\text{if } \mathcal{F} \text{ is reflexive (for all } w \in W,\ w\,R\,w\text{), then } \mathcal{F} \models \Box\varphi \to \varphi.$$

Correspondence

Fact. For any frame $\mathcal{F} = \langle W, R \rangle$,

$$\mathcal{F} \text{ is reflexive (for all } w \in W,\ w\, R\, w) \text{ if, and only if, } \mathcal{F} \models \Box\varphi \rightarrow \varphi.$$

Modal Formula	Corresponding Property
$\Box\varphi \to \varphi$	Reflexive: for all w, $w\,R\,w$
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$\Box\varphi \to \Box\Box\varphi$	Transitive: for all w, v, x, if $w\,R\,v$ and $v\,R\,x$, then $w\,R\,x$
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  Yes, but it only works for certain formulas (The Sahlqvist Theorem)
  SQEMA: https://store.fmi.uni-sofia.bg/fmi/logic/sqema/sqema_gwt_20180317_2/K45/SQEMA.html