# Introduction to Modal Logic

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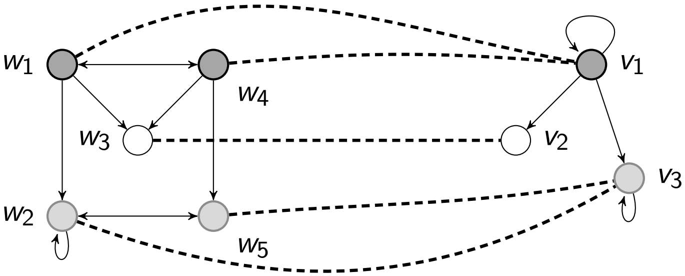
# Bisimulation

A bisimulation between $\mathcal{M} = \langle W, R, V \rangle$ and $\mathcal{M}' = \langle W', R', V' \rangle$ is a non-empty binary relation $Z \subseteq W \times W'$ such that whenever $wZw'$:

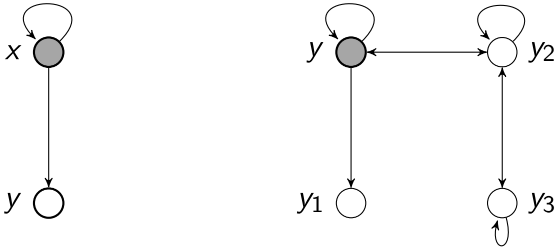
**Atomic harmony:** for each $p \in \mathrm{At}$, $w \in V(p)$ iff $w' \in V'(p)$
**Zig:** if $wRv$, then $\exists v' \in W'$ such that $vZv'$ and $w'R'v'$
**Zag:** if $w'R'v'$ then $\exists v \in W$ such that $vZv'$ and $wRv$

We write $\mathcal{M}, w \underline{\leftrightarrow} \mathcal{M}', w'$ if there is a $Z$ such that $wZw'$.

# Example of a Bisimulation




$$\mathcal{M}, w_1 \underline{\leftrightarrow} \mathcal{M}', v_1$$




It is not the case that $\mathcal{M}, x \underline{\leftrightarrow} \mathcal{M}', y$

$$\mathcal{M}, x \models \Box(\Box\bot \vee \Diamond\Box\bot) \qquad \mathcal{M}', y \not\models \Box(\Box\bot \vee \Diamond\Box\bot)$$

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What about the converse? If two states are modally equivalent, does that imply that they states must be bisimilar?

- In general, it is not true that modally equivalent states are bisimular. That is, there are pointed models $\mathcal{M}, w$ and $\mathcal{M}', w'$ such that $\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w'$, but it is not the case that $\mathcal{M}, w \underline{\leftrightarrow} \mathcal{M}', w'$

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- **Lemma** On finite models, if $\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w'$ then $\mathcal{M}, w \underline{\leftrightarrow} \mathcal{M}', w'$.
- The above result can be generalized: On **image finite models** or $m$-**saturated models**, if $\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w'$ then $\mathcal{M}, w \underline{\leftrightarrow} \mathcal{M}', w'$.

# From truth in a model to validity on a frame

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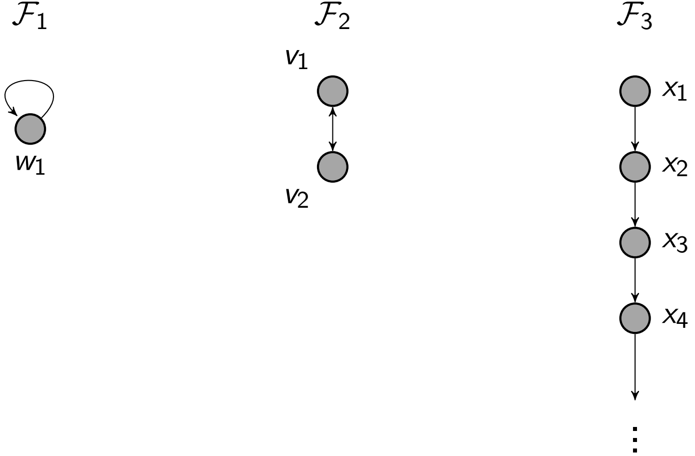
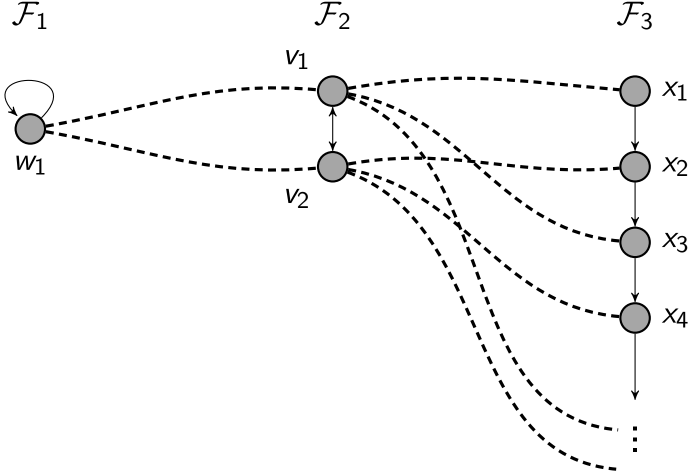
Suppose that $\mathcal{F} = \langle W, R\rangle$ is a frame. A **model based on** $\mathcal{F}$ is a tuple $\langle W, R, V\rangle$ where $V : \mathrm{At} \to \wp(W)$.

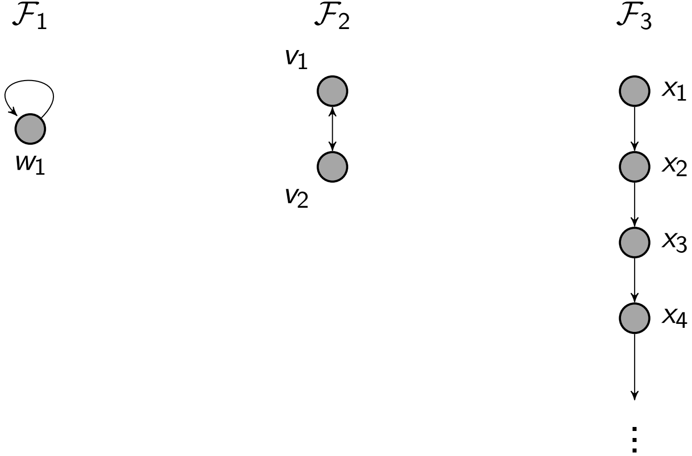
We sometimes write $\langle \mathcal{F}, V\rangle$ for the model based on $\mathcal{F}$.



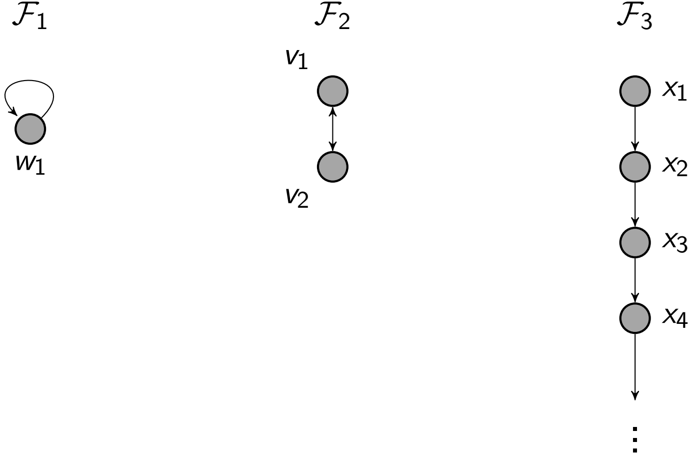
$\mathcal{F}_1$
$\mathcal{F}_2$
$\mathcal{F}_3$
$w_1$
$v_1$
$v_2$
$x_1$
$x_2$
$x_3$
$x_4$




There are valuations $V_1$, $V_2$ and $V_3$ such that $\langle \mathcal{F}_1, V_1 \rangle \underline{\leftrightarrow} \langle \mathcal{F}_2, V_2 \rangle \underline{\leftrightarrow} \langle \mathcal{F}_3, V_3 \rangle$




Can you find a valuation $V_1$ such that $\langle \mathcal{F}_1, V_1 \rangle, w_1 \not\models \Box p \rightarrow p$?
Can you find a valuation $V_2$ such that $\langle \mathcal{F}_2, V_2 \rangle, v_1 \not\models \Box p \rightarrow p$?

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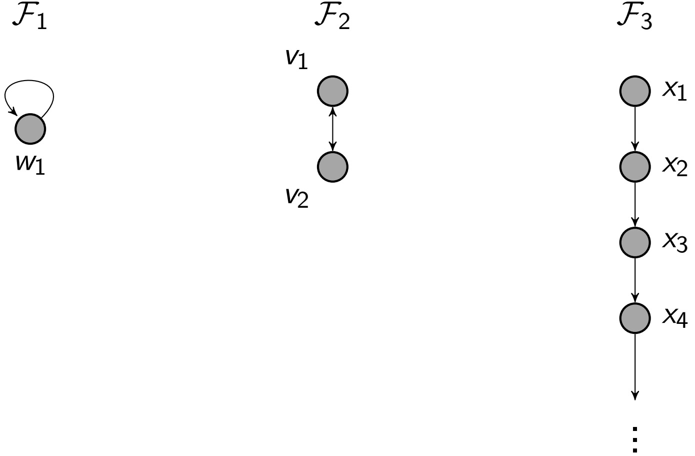


Can you find a valuation $V_2$ such that $\langle \mathcal{F}_2, V_2 \rangle, v_1 \not\models p \rightarrow \Box\Diamond p$?

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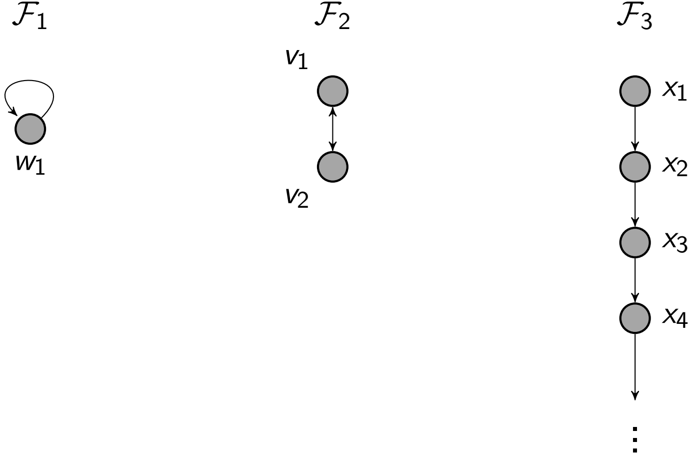


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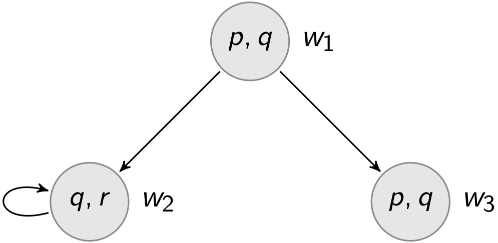
Valid in a class F of frames:

$\models_{\mathsf{F}} \varphi$: for all $\mathcal{F} \in \mathsf{F}$, $\mathcal{F} \models \varphi$

# Model validity




$$\mathcal{M} \models \Box q$$

validity on a model is *not* closed under substitution ($\mathcal{M} \not\models \Box p$)

# Frame validity

Some frame validities:

- $\Box\top$
- $\Box p \leftrightarrow \neg\Diamond\neg p$
- $(\Box p \wedge \Box q) \leftrightarrow \Box(p \wedge q)$
- $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$

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Some frame non-validities:

- $\Box p \vee \Box\neg p$ (compare with the validity $\Box p \vee \neg\Box p$)
- $(\Diamond p \wedge \Diamond q) \rightarrow \Diamond(p \wedge q)$
- $\Box(p \vee q) \rightarrow (\Box p \vee \Box q)$
- $\Box p \rightarrow p$
- $\Box p \rightarrow \Diamond p$