Introduction to Modal Logic

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Definability

Suppose that $\mathcal{M} = \langle W, R, V\rangle$ is a relational model.

$[\![\cdot]\!]_{\mathcal{M}} : \mathcal{L} \to \wp(W)$ defined as $[\![\varphi]\!]_{\mathcal{M}} = \{w \mid \mathcal{M}, w \models \varphi\}$.

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[\![\varphi \wedge \psi]\!]_{\mathcal{M}} & = & [\![\varphi]\!]_{\mathcal{M}} \cap [\![\psi]\!]_{\mathcal{M}} \\
[\![\Box\varphi]\!]_{\mathcal{M}} & = & \{w \mid R(w) \subseteq [\![\varphi]\!]_{\mathcal{M}}\} \\
 & & \text{define } m_R(X) = \{w \mid R(w) \subseteq X\}\text{, so } [\![\Box\varphi]\!]_{\mathcal{M}} = m_R([\![\varphi]\!]_{\mathcal{M}})
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$X \subseteq W$ is **definable** by modal formula if there is some $\varphi \in \mathcal{L}$ such that $X = [\![\varphi]\!]_{\mathcal{M}}$.

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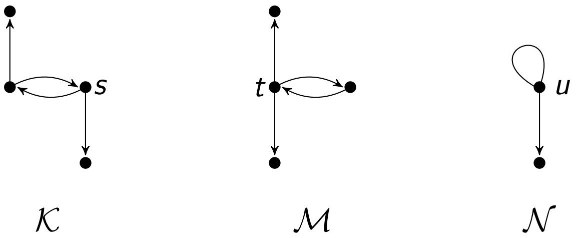
Which pair of states cannot be distinguished by a modal formula?



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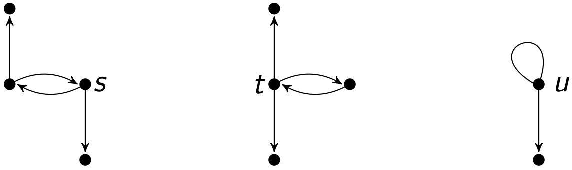

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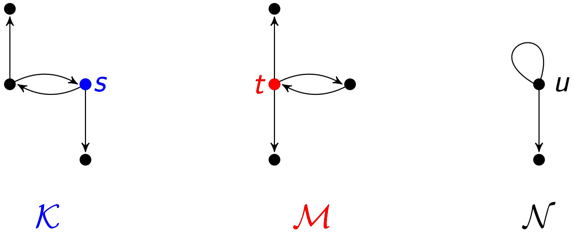

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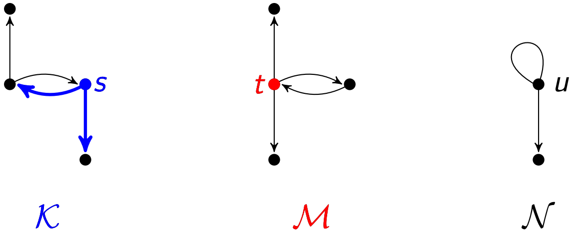

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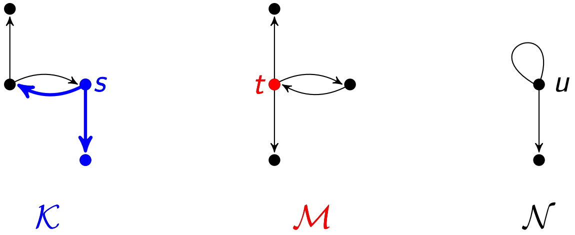

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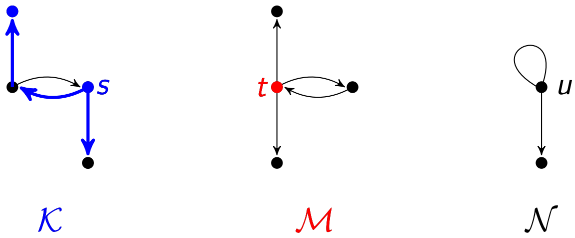

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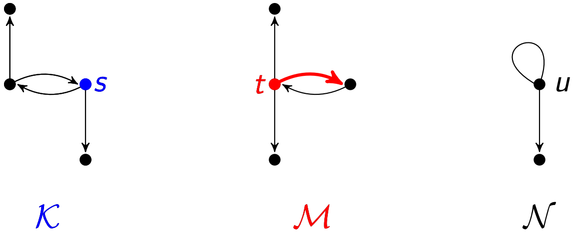

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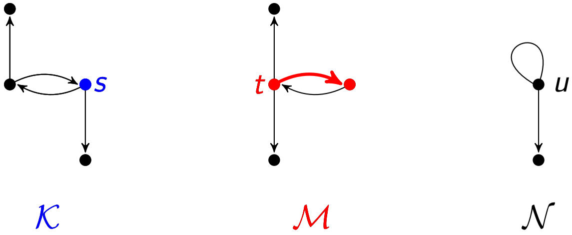

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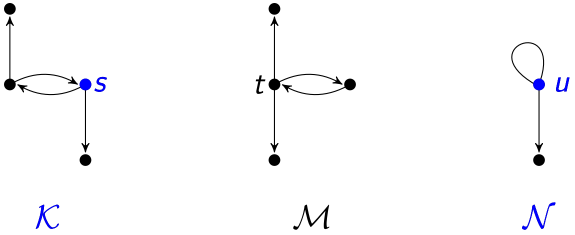

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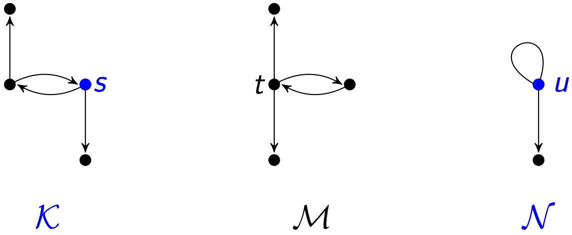

How do you show that $\mathcal{K}$ and $\mathcal{N}$ are *modally equivalent*?

Consider the following modalities:

- $\mathcal{M}, w \models A\varphi$ iff for all $w \in W$, $\mathcal{M}, w \models \varphi$
- $\mathcal{M}, w \models \Diamond^{\leftarrow}\varphi$ iff there is a $v \in W$, vRw and $\mathcal{M}, v \models \varphi$.
- $\mathcal{M}, w \models \Diamond_n\varphi$ iff there are $v_1, \ldots, v_n$ such that for all $1 \leq j \neq k \leq n$, $v_j \neq v_k$, for all $j = 1, \ldots, n$, wRv_j and for all $j = 1, \ldots, n$, $\mathcal{M}, v_j \models \varphi$.

  For instance, $\Diamond_2\varphi$ is true at a state if there are at least two accessible states that satisfy φ.
- $\mathcal{M}, w \models \circlearrowleft$ iff wRw

Are these modalities definable using the basic modal language? Intuitively, the answer is "no", but how do we *prove* this?

Bisimulation

A bisimulation between $\mathcal{M} = \langle W, R, V \rangle$ and $\mathcal{M}' = \langle W', R', V' \rangle$ is a non-empty binary relation $Z \subseteq W \times W'$ such that whenever wZw':

> **Atomic harmony:** for each $p \in \mathrm{At}$, $w \in V(p)$ iff $w' \in V'(p)$
> **Zig:** if wRv, then $\exists v' \in W'$ such that vZv' and $w'R'v'$
> **Zag:** if $w'R'v'$ then $\exists v \in W$ such that vZv' and wRv

We write $\mathcal{M}, w \underline{\leftrightarrow} \mathcal{M}', w'$ if there is a Z such that wZw'.

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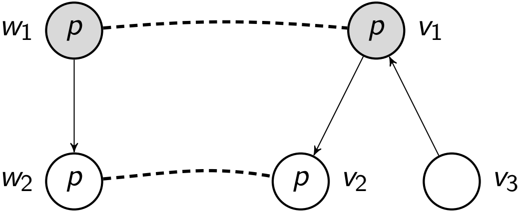

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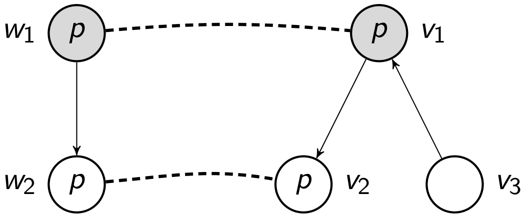

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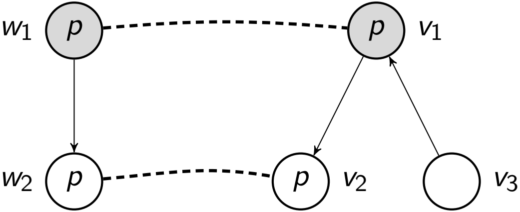

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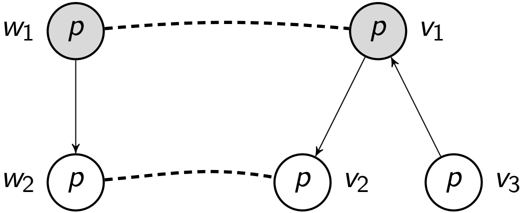

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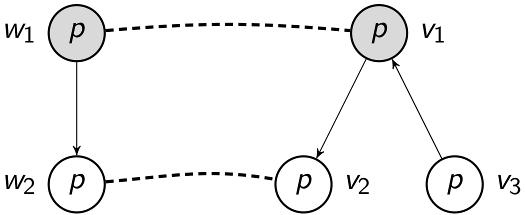

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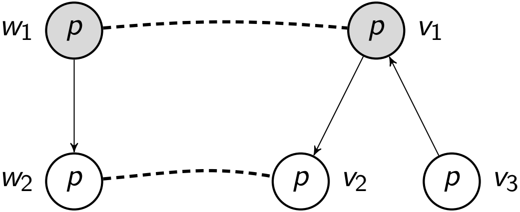

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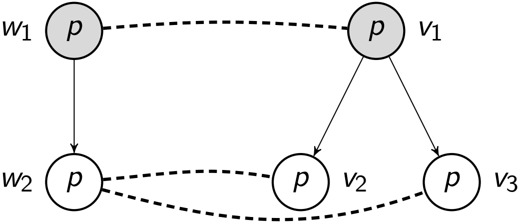

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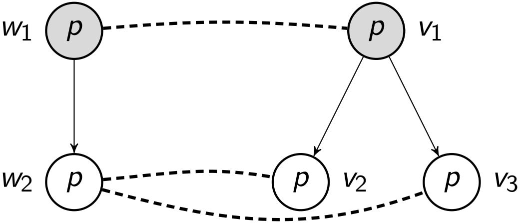

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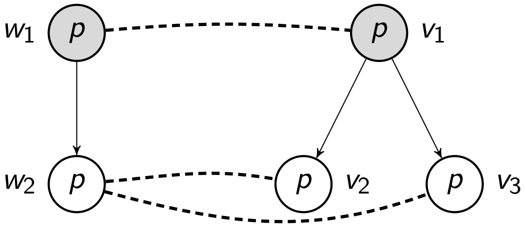

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- In general, it is not true that modally equivalent states are bisimular. That is, there are pointed models $\mathcal{M}, w$ and $\mathcal{M}', w'$ such that $\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w'$, but it is not the case that $\mathcal{M}, w \underline{\leftrightarrow} \mathcal{M}', w'$

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- **Lemma** On finite models, if $\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w'$ then $\mathcal{M}, w \underline{\leftrightarrow} \mathcal{M}', w'$.
- The above result can be generalized: On **image finite models** or m-**saturated models**, if $\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w'$ then $\mathcal{M}, w \underline{\leftrightarrow} \mathcal{M}', w'$.