Introduction to Modal Logic

Eric Pacuit, University of Maryland

September 6, 2023

Temporal Logic

One of the most successful applications of modal logic is in the "logic of time".

Temporal Logic

One of the most successful applications of modal logic is in the "logic of time".

Many variations

- discrete or continuous
- branching or linear
- point based or interval based

V. Goranko and A. Galton. *Temporal Logic*. Stanford Encyclopedia of Philosophy: http://plato.stanford.edu/entries/logic-temporal/.

I. Hodkinson and M. Reynolds. *Temporal Logic*. Handbook of Modal Logic, 2008.

Models of Time

$\mathcal{T} = \langle T, < \rangle$ where

- T is a set of **time points** (or **moments**),
- $< \subseteq T \times T$ is the **precedence relation**: $s < t$ means "time point s precedes time point t (or s occurs earlier than t)" and

Models of Time

$\mathcal{T} = \langle T, < \rangle$ where

- T is a set of **time points** (or **moments**),
- $< \subseteq T \times T$ is the **precedence relation**: $s < t$ means "time point s precedes time point t (or s occurs earlier than t)" and

$<$ is typically assumed to be **irreflexive** and **transitive** (a **strict partial order**).

Models of Time

$\mathcal{T} = \langle T, < \rangle$ where

- T is a set of **time points** (or **moments**),
- $< \subseteq T \times T$ is the **precedence relation**: $s < t$ means "time point s precedes time point t (or s occurs earlier than t)" and

$<$ is typically assumed to be **irreflexive** and **transitive** (a **strict partial order**).

Examples: $\langle \mathbb{N}, < \rangle$, $\langle \mathbb{Z}, < \rangle$, $\langle \mathbb{Q}, < \rangle$, $\langle \mathbb{R}, < \rangle$

Other properties of $<$

- **Linearity**: for all $s, t \in T$, $s < t$ or $s = t$ of $t < s$
- **Past-linear**: for all $s, x, y \in T$, if $x < s$ and $y < s$, then either $x < y$ or $x = y$ or $y < x$
- **Denseness** for all $s, t \in T$, if $s < t$ then there is a $z \in T$ such that $s < z$ and $z < t$
- **Discreteness**: for all $s, t \in T$, if $s < t$ then there is a z such that ($s < z$ and there is no u such that $s < u$ and $u < z$)

Branching Time

Each moment $t \in T$ can be decided into the $Past(t) = \{s \in T \mid s < t\}$ and the $Future(t) = \{s \in T \mid t < s\}$

Typically, it is assumed that the past is linear, but the future may be branching.

Branching Time

Each moment $t \in T$ can be decided into the $Past(t) = \{s \in T \mid s < t\}$ and the $Future(t) = \{s \in T \mid t < s\}$

Typically, it is assumed that the past is linear, but the future may be branching.

$F\varphi$: "it will be the case that φ"

φ will be the case "in the case in the actual course of events" or "no matter what course of events"

Branching Time Logics

A **branch** b in $\langle T, < \rangle$ is a maximal linearly ordered subset of T

$s \in T$ is **on a branch** b **of** T provided $s \in b$ (we also say "b is a branch going through t").

Temporal Logics

Temporal Logics

- *Linear Time Temporal Logic*: Reasoning about computation paths:

  $F\varphi$: φ is true some time in *the* future.

A. Pnuelli. *A Temporal Logic of Programs*. in *Proc. 18th IEEE Symposium on Foundations of Computer Science* (1977).

Temporal Logics

- *Linear Time Temporal Logic*: Reasoning about computation paths:

  $F\varphi$: φ is true some time in *the* future.

A. Pnuelli. *A Temporal Logic of Programs*. in *Proc. 18th IEEE Symposium on Foundations of Computer Science* (1977).

- *Branching Time Temporal Logic*: Allows quantification over paths:

  $\exists F\varphi$: there is a path in which φ is eventually true.

E. M. Clarke and E. A. Emerson. *Design and Synthesis of Synchronization Skeletons using Branching-time Temporal-logic Specifications*. In *Proceedings Workshop on Logic of Programs*, LNCS (1981).

Interval Values

J. Allen and G. Ferguson. *Actions and Events in Interval Temporal Logics*. Journal of Logic and Computation, 1994.

J. Halpern and Y. Shoham. *A Propositional Modal Logic of Time Intervals*. Journal of the ACM, 38:4, pp. 935 - 962, 1991.

J. van Benthem. *Logics of Time*. Kluwer, 1991.

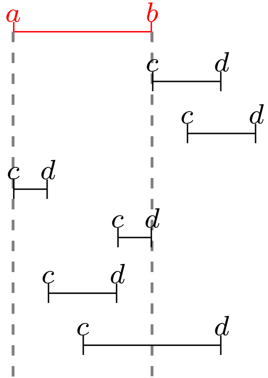
Interval Temporal Logics

Let $\mathcal{T} = \langle T, < \rangle$ be a frame and $I(\mathcal{T}) = \{[a, b] \mid a, b \in T \text{ and } a \leq b\}$ be the set of intervals over T

Interval-based relational structure: $\langle I(\mathcal{T}), \{R_X\} \rangle$ where $R_X \subseteq I(\mathcal{T}) \times I(\mathcal{T})$.

Interval Temporal Logics

$\langle A \rangle$	$[a,b] R_A [c,d] \Leftrightarrow b = c$
$\langle L \rangle$	$[a,b] R_L [c,d] \Leftrightarrow b < c$
$\langle B \rangle$	$[a,b] R_B [c,d] \Leftrightarrow a = c, d < b$
$\langle E \rangle$	$[a,b] R_E [c,d] \Leftrightarrow b = d, a < c$
$\langle D \rangle$	$[a,b] R_D [c,d] \Leftrightarrow a < c, d < b$
$\langle O \rangle$	$[a,b] R_O [c,d] \Leftrightarrow a < c < b < d$



Propositional Modal Language

Language: Let At be a set of atomic propositions. The set of propositional modal formulas, denoted $\mathcal{L}(\mathrm{At})$, is the smallest set of formulas generated by the following grammar:

$$p \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid \Diamond\varphi$$

where $p \in \mathrm{At}$.

Frame: $\langle W, R \rangle$, where $W \neq \emptyset$ and $R \subseteq W \times W$

Frame: $\langle W, R \rangle$, where $W \neq \emptyset$ and $R \subseteq W \times W$

Model: Suppose that $\mathcal{F} = \langle W, R \rangle$ is a frame. The tuple $\langle W, R, V \rangle$ is a **model based on** $\mathcal{F}$ where $V : \mathrm{At} \to \wp(W)$ is a **valuation function**.

- $w \in V(p)$ means that p is true at w.

Frame: $\langle W, R\rangle$, where $W \neq \emptyset$ and $R \subseteq W \times W$

Model: Suppose that $\mathcal{F} = \langle W, R\rangle$ is a frame. The tuple $\langle W, R, V\rangle$ is a **model based on** $\mathcal{F}$ where $V : \mathrm{At} \to \wp(W)$ is a **valuation function**.

- $w \in V(p)$ means that p is true at w.

Pointed Model Suppose that $\mathcal{M} = \langle W, R, V\rangle$ is a model. If $w \in W$, then $(\mathcal{M}, w)$ is called a **pointed model**.

Truth of Modal Formulas

Suppose that $\mathcal{M} = \langle W, R, V \rangle$ is a model. Truth of a modal formula $\varphi \in \mathcal{L}(\mathsf{At})$ at a state w in $\mathcal{M}$, denoted $\mathcal{M}, w \models \varphi$, is defined as follows:

- $\mathcal{M}, w \models p$ iff $w \in V(p)$ (where $p \in \mathsf{At}$)

Truth of Modal Formulas

Suppose that $\mathcal{M} = \langle W, R, V \rangle$ is a model. Truth of a modal formula $\varphi \in \mathcal{L}(\mathsf{At})$ at a state w in $\mathcal{M}$, denoted $\mathcal{M}, w \models \varphi$, is defined as follows:

- $\mathcal{M}, w \models p$ iff $w \in V(p)$ (where $p \in \mathsf{At}$)
- $\mathcal{M}, w \not\models \bot$
- $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$
- $\mathcal{M}, w \models \varphi \vee \psi$ iff $\mathcal{M}, w \models \varphi$ or $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \varphi \to \psi$ iff if $\mathcal{M}, w \models \varphi$, then $\mathcal{M}, w \models \psi$
  iff either $\mathcal{M}, w \not\models \varphi$ or $\mathcal{M}, w \models \psi$

Truth of Modal Formulas

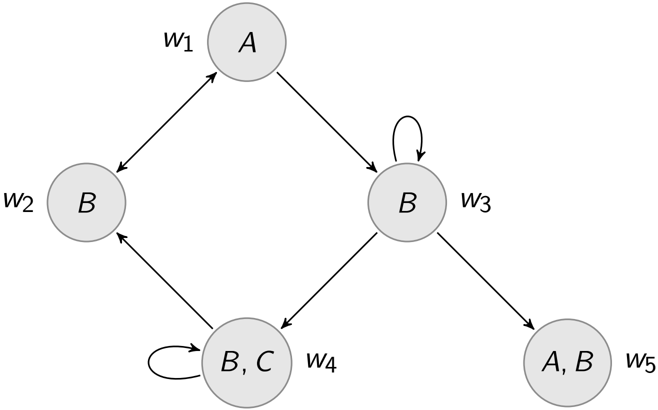
Suppose that $\mathcal{M} = \langle W, R, V \rangle$ is a model. Truth of a modal formula $\varphi \in \mathcal{L}(\mathsf{At})$ at a state w in $\mathcal{M}$, denoted $\mathcal{M}, w \models \varphi$, is defined as follows:

- $\mathcal{M}, w \models p$ iff $w \in V(p)$ (where $p \in \mathsf{At}$)
- $\mathcal{M}, w \not\models \bot$
- $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$
- $\mathcal{M}, w \models \varphi \vee \psi$ iff $\mathcal{M}, w \models \varphi$ or $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \varphi \to \psi$ iff if $\mathcal{M}, w \models \varphi$, then $\mathcal{M}, w \models \psi$
  iff either $\mathcal{M}, w \not\models \varphi$ or $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \Diamond\varphi$ iff there is a $v \in W$ such that wRv and $\mathcal{M}, v \models \varphi$
- $\mathcal{M}, w \models \Box\varphi$ iff for all $v \in W$, if wRv then $\mathcal{M}, v \models \varphi$

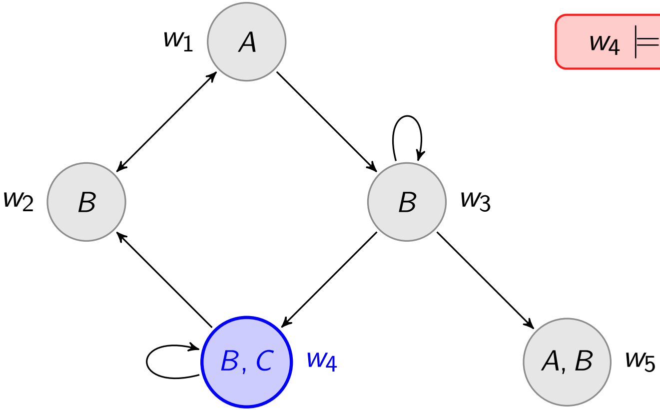
Truth of Modal Formulas

- $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \varphi \rightarrow \psi$ iff if $\mathcal{M}, w \models \varphi$, then $\mathcal{M}, w \models \psi$
  iff either $\mathcal{M}, w \not\models \varphi$ or $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \Box\varphi$ iff for all $v \in W$, if wRv then $\mathcal{M}, v \models \varphi$

Example

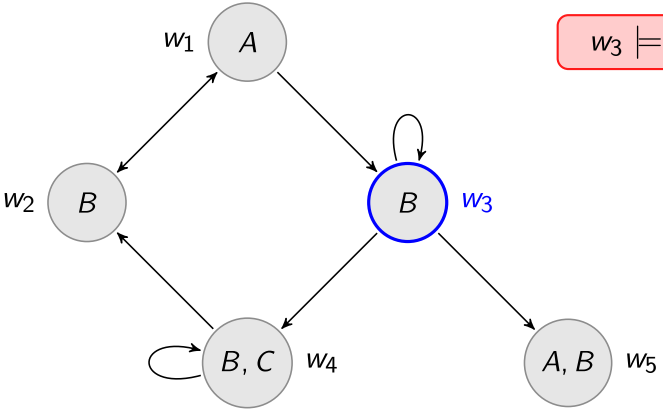


Example



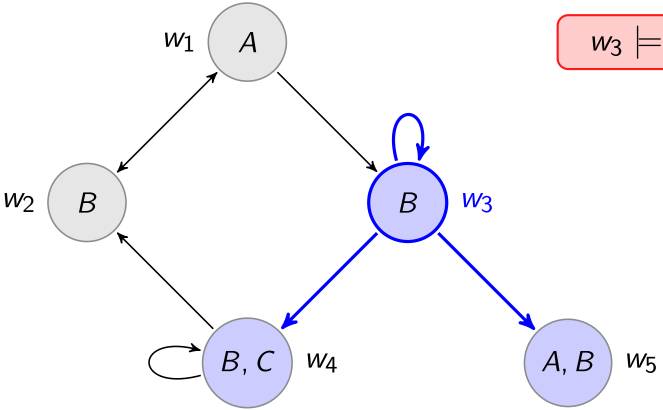
$w_4 \models B \wedge C$

Example



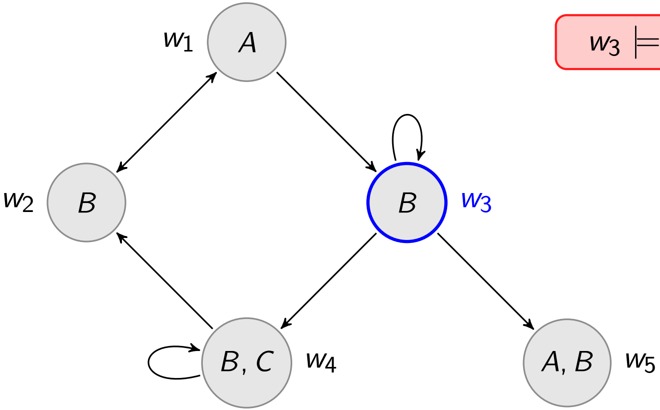
$w_3 \models \Box B$

Example



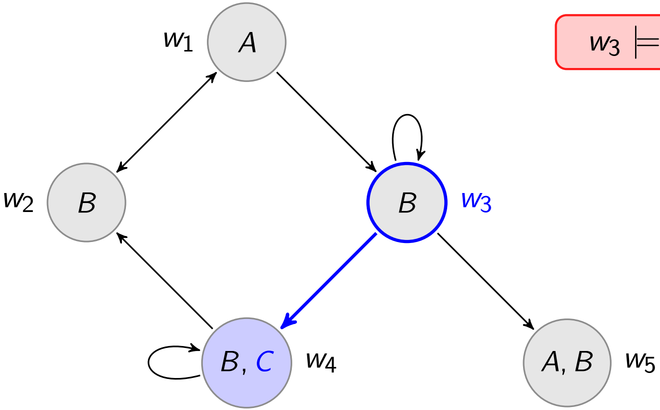
$w_3 \models \Box B$

Example



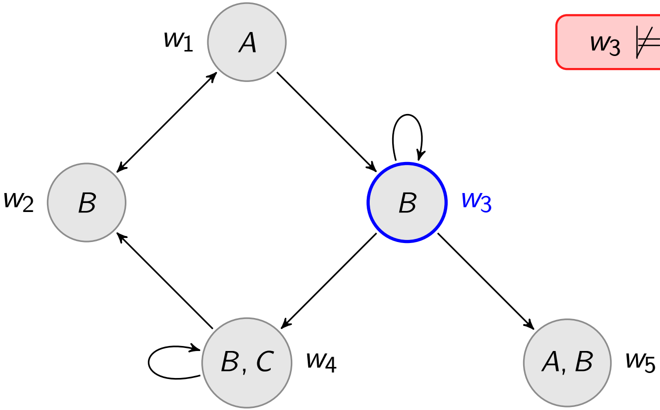
$w_3 \models \Diamond C$

Example



$w_3 \models \Diamond C$

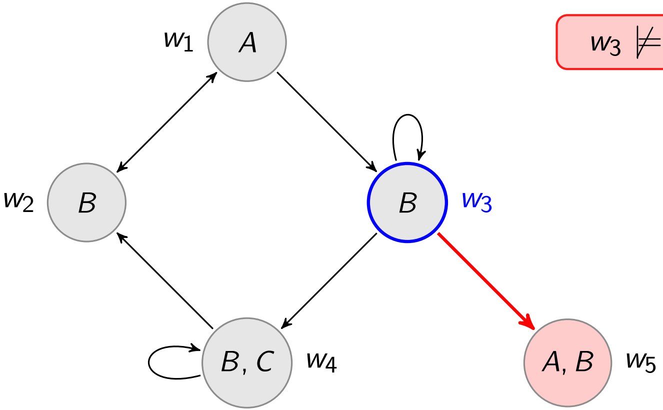
Example



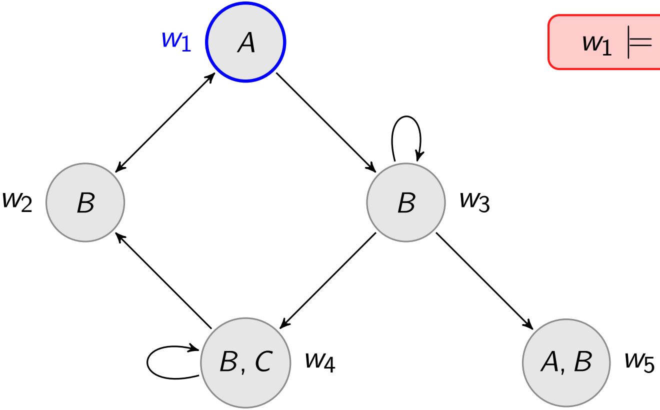
$w_3 \not\models \Box C$

Example

$w_3 \not\models \Box C$

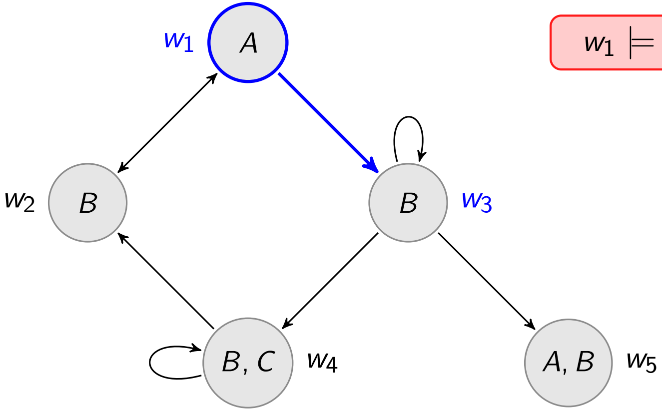


Example



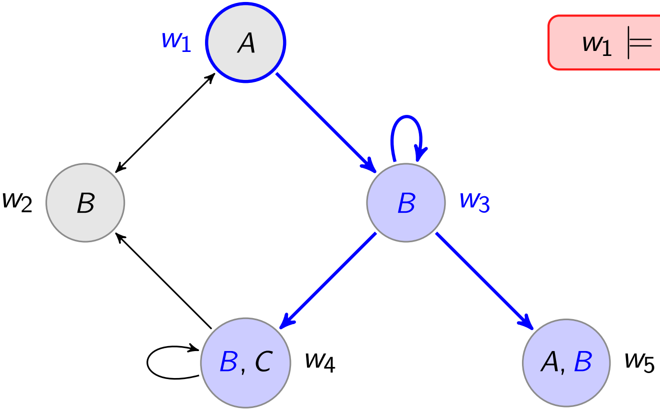
$w_1 \models \Diamond\Box B$

Example



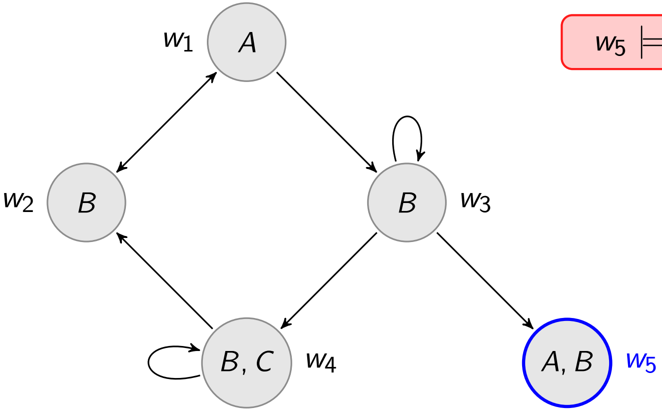
$w_1 \models \Diamond\Box B$

Example



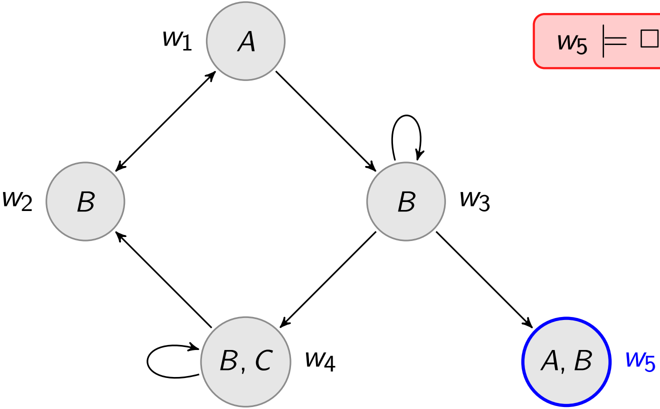
$w_1 \models \Diamond \Box B$

Example



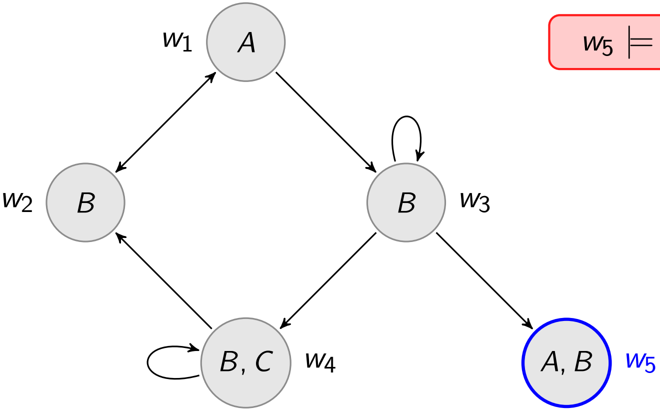
$w_5 \models \Box C$

Example

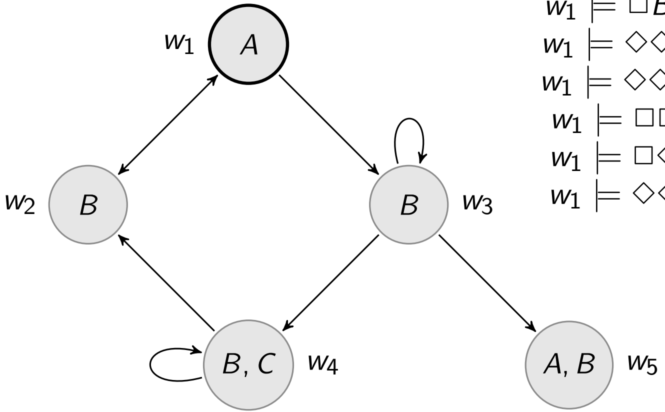


$w_5 \models \Box(B \land \neg B)$

Example



$w_5 \models \neg \Diamond B$



$w_1 \models \Box B \wedge B?$

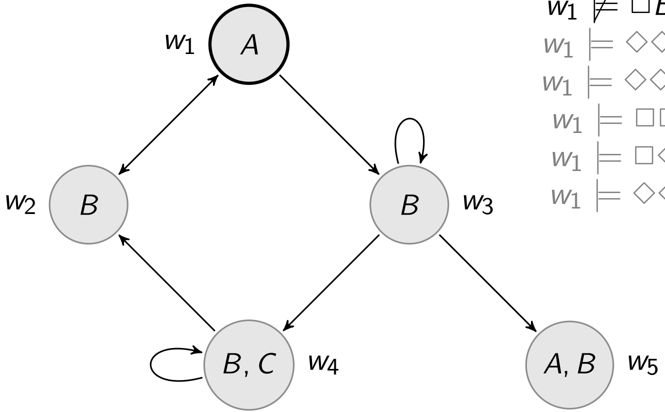
$w_1 \models \Diamond\Diamond B?$

$w_1 \models \Diamond\Diamond\Diamond B?$

$w_1 \models \Box\Box B?$

$w_1 \models \Box\Diamond C?$

$w_1 \models \Diamond\Diamond C?$



$w_1 \not\models \Box B \land B$

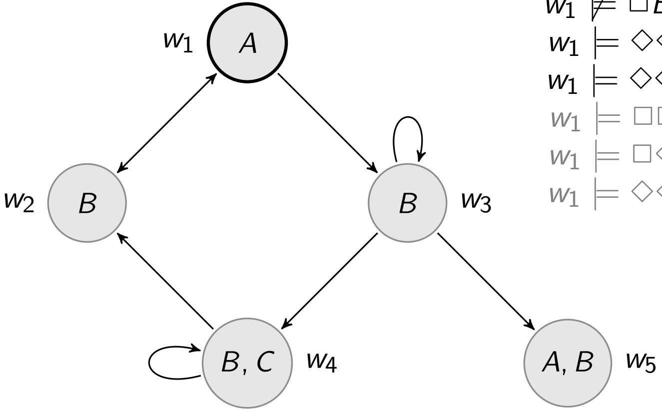
$w_1 \models \Diamond\Diamond B?$

$w_1 \models \Diamond\Diamond\Diamond B?$

$w_1 \models \Box\Box B?$

$w_1 \models \Box\Diamond C?$

$w_1 \models \Diamond\Diamond C?$



$w_1 \not\models \Box B \wedge B$

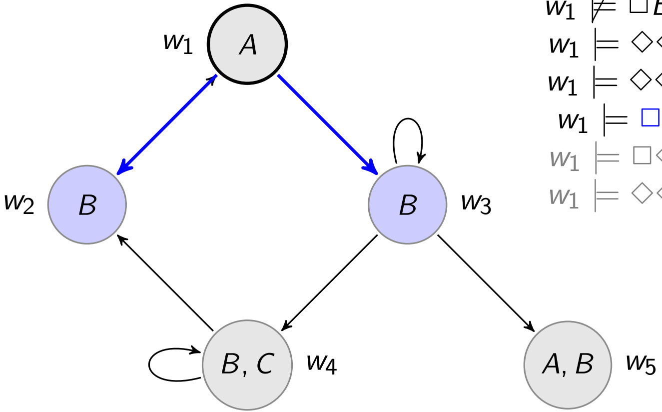
$w_1 \models \Diamond\Diamond B$

$w_1 \models \Diamond\Diamond\Diamond B$

$w_1 \models \Box\Box B$?

$w_1 \models \Box\Diamond C$?

$w_1 \models \Diamond\Diamond C$?



$w_1 \not\models \Box B \wedge B$

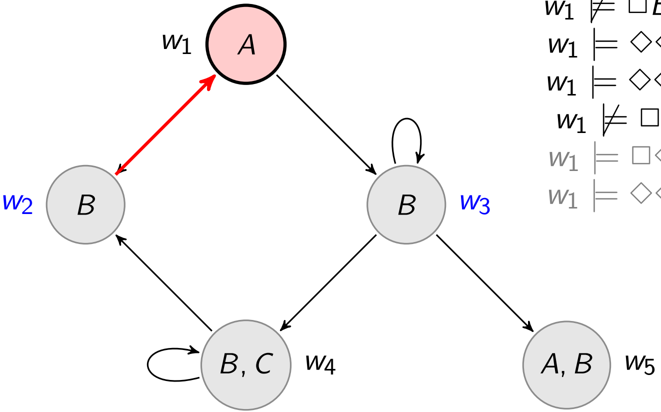
$w_1 \models \Diamond\Diamond B$

$w_1 \models \Diamond\Diamond\Diamond B$

$w_1 \models \Box\Box B$

$w_1 \models \Box\Diamond C?$

$w_1 \models \Diamond\Diamond C?$



$w_1 \not\models \Box B \wedge B$

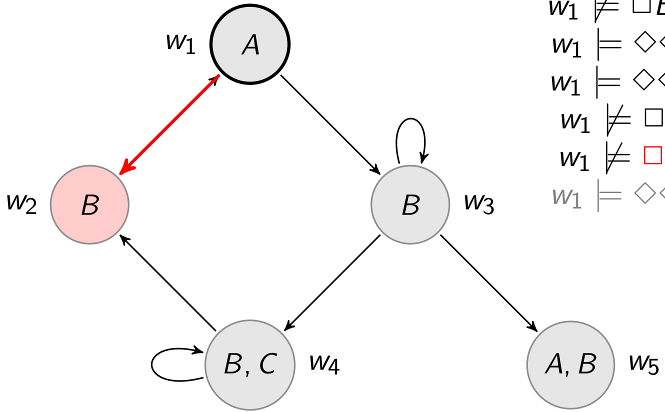
$w_1 \models \Diamond\Diamond B$

$w_1 \models \Diamond\Diamond\Diamond B$

$w_1 \not\models \Box\Box B$

$w_1 \models \Box\Diamond C?$

$w_1 \models \Diamond\Diamond C?$



$w_1 \not\models \Box B \wedge B$

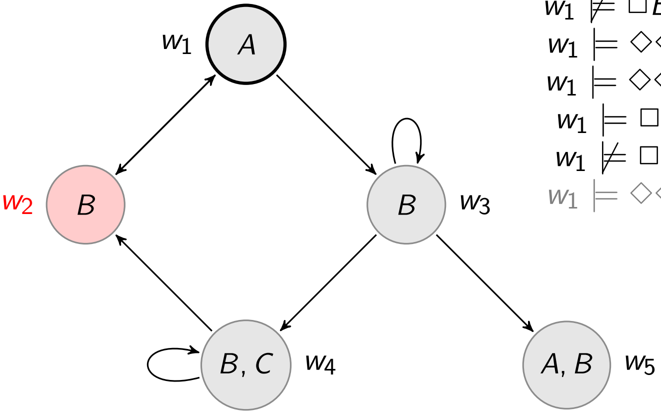
$w_1 \models \Diamond\Diamond B$

$w_1 \models \Diamond\Diamond\Diamond B$

$w_1 \not\models \Box\Box B$

$w_1 \not\models \Box\Diamond C$

$w_1 \models \Diamond\Diamond C?$



$w_1 \not\models \Box B \wedge B$

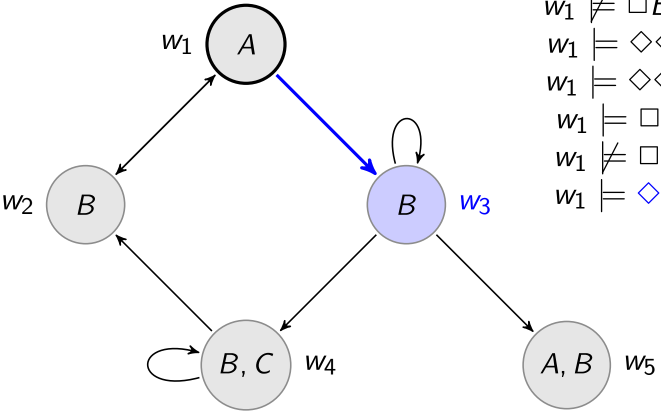
$w_1 \models \Diamond\Diamond B$

$w_1 \models \Diamond\Diamond\Diamond B$

$w_1 \models \Box\Box B$

$w_1 \not\models \Box\Diamond C$

$w_1 \models \Diamond\Diamond C$?



$w_1 \not\models \Box B \wedge B$

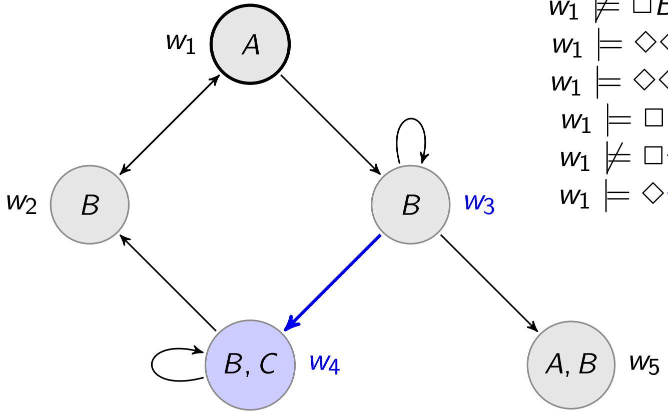
$w_1 \models \Diamond\Diamond B$

$w_1 \models \Diamond\Diamond\Diamond B$

$w_1 \models \Box\Box B$

$w_1 \not\models \Box\Diamond C$

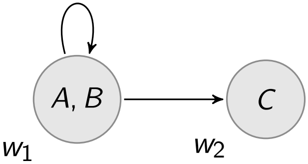
$w_1 \models \Diamond\Diamond C$



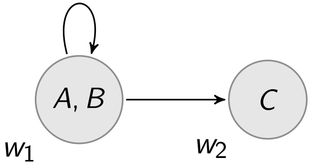
$$w_1 \not\models \Box B \wedge B$$
$$w_1 \models \Diamond\Diamond B$$
$$w_1 \models \Diamond\Diamond\Diamond B$$
$$w_1 \models \Box\Box B$$
$$w_1 \not\models \Box\Diamond C$$
$$w_1 \models \Diamond\Diamond C$$

We can now see why the two formalizations of Aristotle's Sea Battle Argument are not *equivalent*: $\Box(A \to B)$ is is not equivalent with $A \to \Box B$.

We can now see why the two formalizations of Aristotle's Sea Battle Argument are not *equivalent*: $\Box(A \rightarrow B)$ is is not equivalent with $A \rightarrow \Box B$.



We can now see why the two formalizations of Aristotle's Sea Battle Argument are not *equivalent*: $\Box(A \to B)$ is is not equivalent with $A \to \Box B$.



$$w_1 \models \Box(A \to B) \text{ but } w_1 \not\models A \to \Box B$$

Definability

Suppose that $\mathcal{M} = \langle W, R, V \rangle$ is a relational model.

$[\![\cdot]\!]_{\mathcal{M}} : \mathcal{L} \to \wp(W)$ defined as $[\![\varphi]\!]_{\mathcal{M}} = \{w \mid \mathcal{M}, w \models \varphi\}$.

$$\begin{array}{rcl} [\![p]\!]_{\mathcal{M}} & = & V(p) \\ [\![\neg\varphi]\!]_{\mathcal{M}} & = & W - [\![\varphi]\!]_{\mathcal{M}} \\ [\![\varphi \wedge \psi]\!]_{\mathcal{M}} & = & [\![\varphi]\!]_{\mathcal{M}} \cap [\![\psi]\!]_{\mathcal{M}} \\ [\![\Box\varphi]\!]_{\mathcal{M}} & = & \{w \mid R(w) \subseteq X\} \\ & & \text{define } m_R(X) = \{w \mid R(w) \subseteq X\}\text{, so } [\![\Box\varphi]\!]_{\mathcal{M}} = m_R([\![\varphi]\!]_{\mathcal{M}}) \end{array}$$

Definability

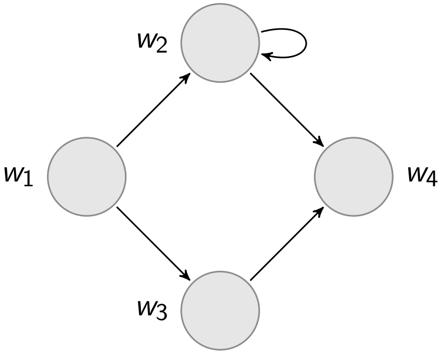
Suppose that $\mathcal{M} = \langle W, R, V \rangle$ is a relational model.

$[\![\cdot]\!]_{\mathcal{M}} : \mathcal{L} \to \wp(W)$ defined as $[\![\varphi]\!]_{\mathcal{M}} = \{w \mid \mathcal{M}, w \models \varphi\}$.

$$
\begin{array}{rcl}
[\![p]\!]_{\mathcal{M}} & = & V(p) \\
[\![\neg\varphi]\!]_{\mathcal{M}} & = & W - [\![\varphi]\!]_{\mathcal{M}} \\
[\![\varphi \wedge \psi]\!]_{\mathcal{M}} & = & [\![\varphi]\!]_{\mathcal{M}} \cap [\![\psi]\!]_{\mathcal{M}} \\
[\![\Box\varphi]\!]_{\mathcal{M}} & = & \{w \mid R(w) \subseteq X\} \\
 & & \text{define } m_R(X) = \{w \mid R(w) \subseteq X\}, \text{ so } [\![\Box\varphi]\!]_{\mathcal{M}} = m_R([\![\varphi]\!]_{\mathcal{M}})
\end{array}
$$

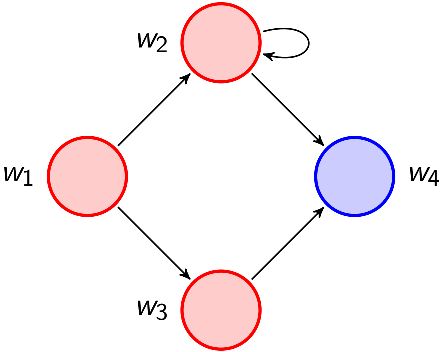
$X \subseteq W$ is **definable** by modal formula if there is some $\varphi \in \mathcal{L}$ such that $X = [\![\varphi]\!]_{\mathcal{M}}$.

Defining States



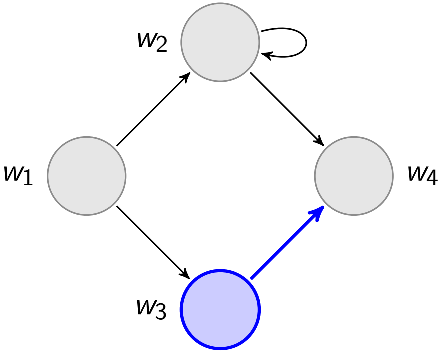
- $\{w_4\} =$
- $\{w_3\} =$
- $\{w_2\} =$
- $\{w_1\} =$

Defining States



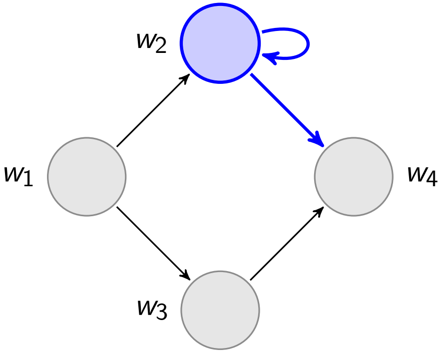
- $\{w_4\} = [\![\Box\bot]\!]$
- $\{w_3\} =$
- $\{w_2\} =$
- $\{w_1\} =$

Defining States



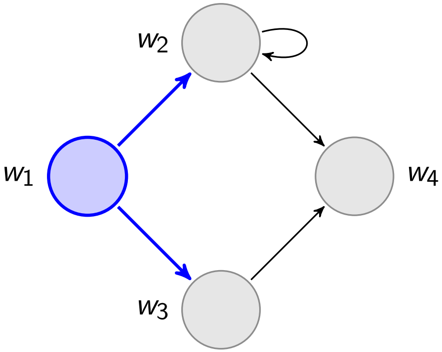
- $\{w_4\} = [\![\Box\bot]\!]$
- $\{w_3\} = [\![\Diamond\Box\bot \wedge \Box\Box\bot]\!]$
- $\{w_2\} =$
- $\{w_1\} =$

Defining States



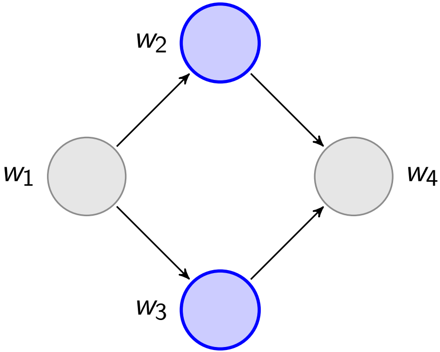
- $\{w_4\} = [\![\Box\bot]\!]$
- $\{w_3\} = [\![\Diamond\Box\bot \land \Box\Box\bot]\!]$
- $\{w_2\} = [\![\Diamond\Box\bot \land \Diamond\Diamond\top]\!]$
- $\{w_1\} =$

Defining States



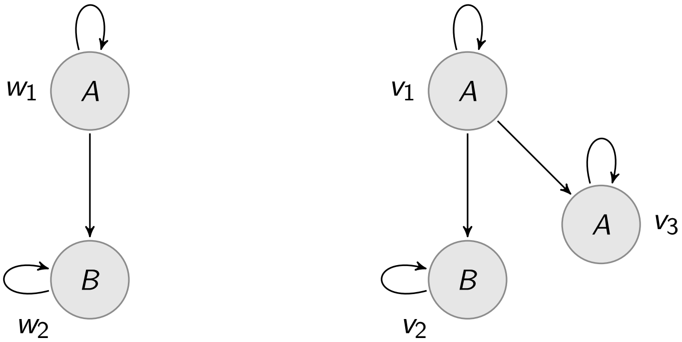
- $\{w_4\} = [\![\Box\bot]\!]$
- $\{w_3\} = [\![\Diamond\Box\bot \wedge \Box\Box\bot]\!]$
- $\{w_2\} = [\![\Diamond\Box\bot \wedge \Diamond\Diamond\top]\!]$
- $\{w_1\} = [\![\Diamond(\Diamond\Box\bot \wedge \Box\Box\bot)]\!]$

Defining States



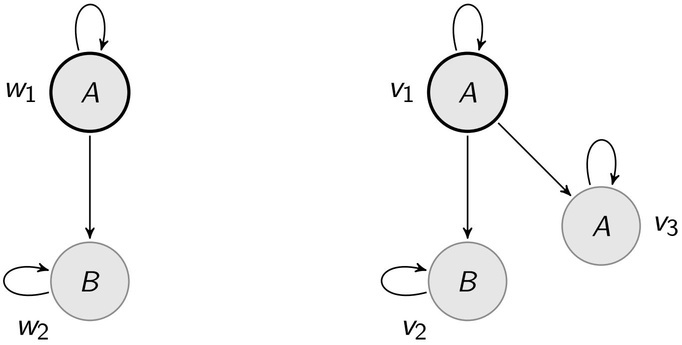
- $\{w_4\} = [\![\Box\bot]\!]$
- $\{w_2, w_3\} = [\![\Diamond\Box\bot \wedge \Box\Box\bot]\!]$
- $\{w_1\} = [\![\Diamond(\Diamond\Box\bot \wedge \Box\Box\bot)]\!]$

Distinguishing States



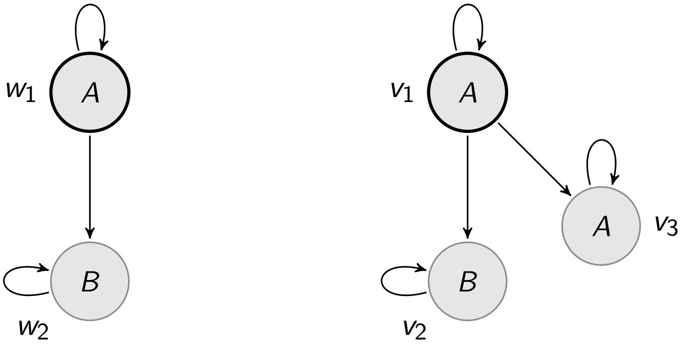
What is the difference between states w_1 and v_1?

Distinguishing States



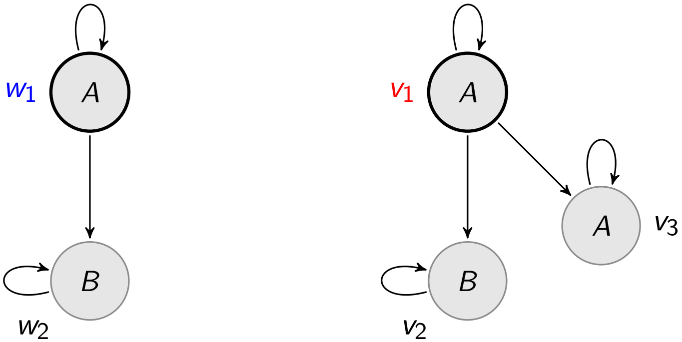
What is the difference between states w_1 and v_1?

Distinguishing States



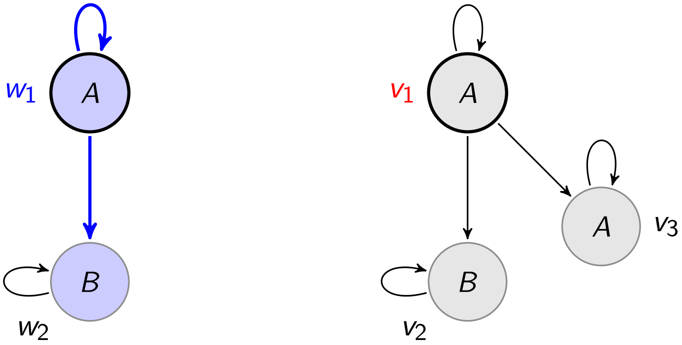
Is there a modal formula true at w_1 but not at v_1?

Distinguishing States



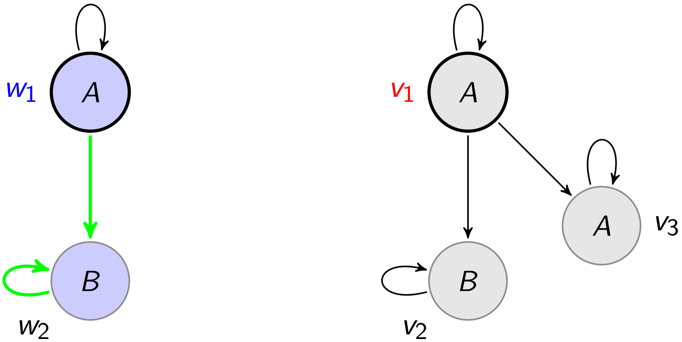
$w_1 \models \Box\Diamond\neg A$ but $v_1 \not\models \Box\Diamond\neg A$.

Distinguishing States



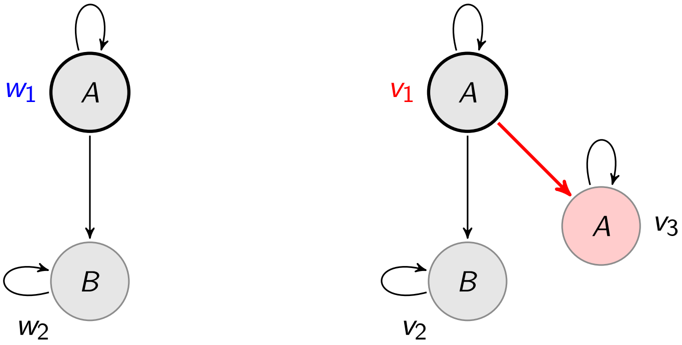
$w_1 \models \Box\Diamond\neg A$ but $v_1 \not\models \Box\Diamond\neg A$.

Distinguishing States



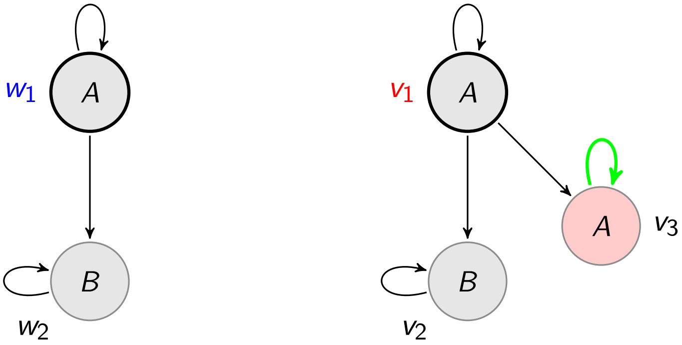
$w_1 \models \Box\Diamond\neg A$ but $v_1 \not\models \Box\Diamond\neg A$.

Distinguishing States



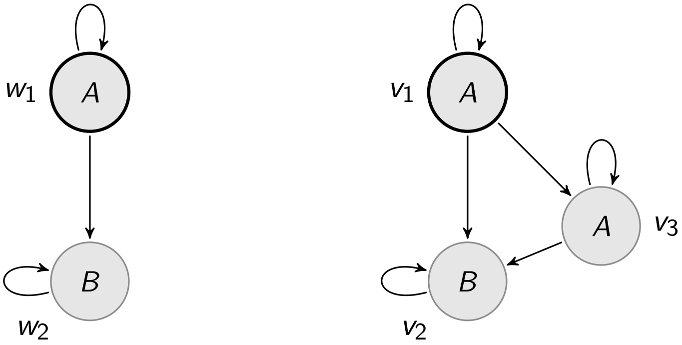
$$w_1 \models \Box\Diamond\neg A \text{ but } v_1 \not\models \Box\Diamond\neg A.$$

Distinguishing States



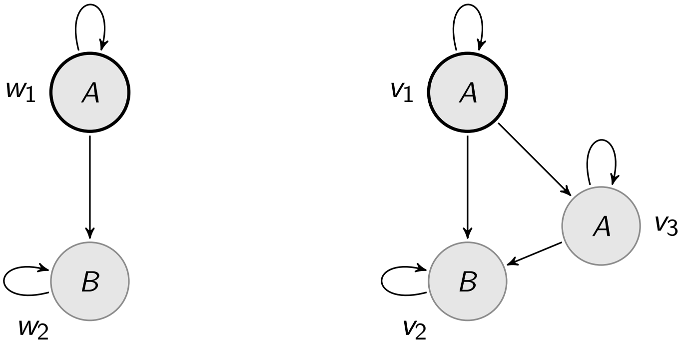
$$w_1 \models \Box\Diamond\neg A \text{ but } v_1 \not\models \Box\Diamond\neg A.$$

Distinguishing States



What about now? Is there a modal formula true at w_1 but not v_1?

Distinguishing States



No modal formula can distinguish w_1 and v_1!

φ is **satisfiable** means that there is a model $\mathcal{M} = \langle W, R, V \rangle$ and $w \in W$ such that $\mathcal{M}, w \models \varphi$.

Bisimulation

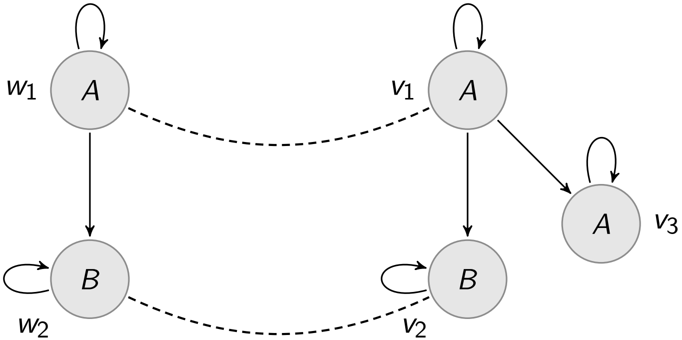
A bisimulation between $\mathcal{M} = \langle W, R, V \rangle$ and $\mathcal{M}' = \langle W', R', V' \rangle$ is a non-empty binary relation $Z \subseteq W \times W'$ such that whenever wZw':

Atomic harmony: for each $p \in \text{At}$, $w \in V(p)$ iff $w' \in V'(p)$
Zig: if wRv, then $\exists v' \in W'$ such that vZv' and $w'R'v'$
Zag: if $w'R'v'$ then $\exists v \in W$ such that vZv' and wRv

Bisimulation

A bisimulation between $\mathcal{M} = \langle W, R, V \rangle$ and $\mathcal{M}' = \langle W', R', V' \rangle$ is a non-empty binary relation $Z \subseteq W \times W'$ such that whenever wZw':

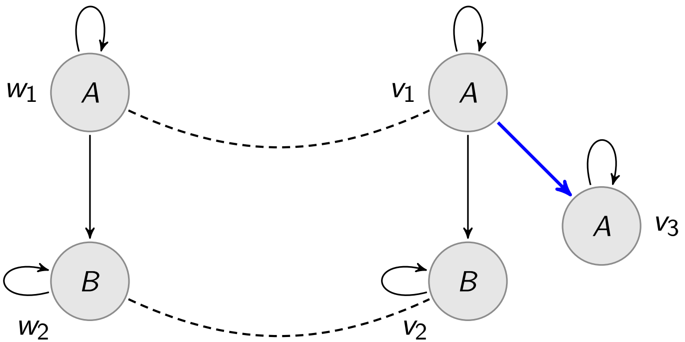
Atomic harmony: for each $p \in \mathsf{At}$, $w \in V(p)$ iff $w' \in V'(p)$
Zig: if wRv, then $\exists v' \in W'$ such that vZv' and $w'R'v'$
Zag: if $w'R'v'$ then $\exists v \in W$ such that vZv' and wRv



Bisimulation

A bisimulation between $\mathcal{M} = \langle W, R, V \rangle$ and $\mathcal{M}' = \langle W', R', V' \rangle$ is a non-empty binary relation $Z \subseteq W \times W'$ such that whenever wZw':

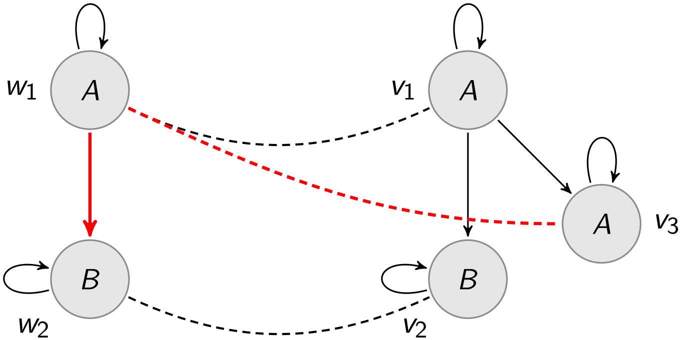
Atomic harmony: for each $p \in \mathsf{At}$, $w \in V(p)$ iff $w' \in V'(p)$
Zig: if wRv, then $\exists v' \in W'$ such that vZv' and $w'R'v'$
Zag: if $w'R'v'$ then $\exists v \in W$ such that vZv' and wRv



Bisimulation

A bisimulation between $\mathcal{M} = \langle W, R, V \rangle$ and $\mathcal{M}' = \langle W', R', V' \rangle$ is a non-empty binary relation $Z \subseteq W \times W'$ such that whenever wZw':

Atomic harmony: for each $p \in \text{At}$, $w \in V(p)$ iff $w' \in V'(p)$
Zig: if wRv, then $\exists v' \in W'$ such that vZv' and $w'R'v'$
Zag: if $w'R'v'$ then $\exists v \in W$ such that vZv' and wRv



Bisimulation

A bisimulation between $\mathcal{M} = \langle W, R, V \rangle$ and $\mathcal{M}' = \langle W', R', V' \rangle$ is a non-empty binary relation $Z \subseteq W \times W'$ such that whenever wZw':

Atomic harmony: for each $p \in \text{At}$, $w \in V(p)$ iff $w' \in V'(p)$
Zig: if wRv, then $\exists v' \in W'$ such that vZv' and $w'R'v'$
Zag: if $w'R'v'$ then $\exists v \in W$ such that vZv' and wRv