First Order Modal Logic

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First Order Modal Language

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A **term** is any variable: $\mathcal{T} = \mathcal{V}$ (we will extend the set of terms later when we add **constants** and **function symbols** to the language)

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A **formula** is constructed as follows

$$\varphi \quad := \quad t_1 = t_2 \mid P(t_1, \ldots, t_n) \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid \Diamond\varphi \mid (\forall x)\varphi \mid (\exists x)\varphi$$

where $P \in \mathit{Pred}$ of arity n, $t_i \in \mathcal{T}$ for $i = 1, \ldots, n$ and $x \in \mathcal{V}$

(Sometimes equality is not in the language)

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A **varying domain Kripke skeleton** is a tuple $\langle W, R, \mathcal{D} \rangle$ where $\langle W, R \rangle$ is a frame and for each $w \in W$, $\mathcal{D}(w)$ is a set (the domain at w). Let the **domain of the model** be $D = \bigcup_{w \in W} \mathcal{D}(w)$.

Substitutions

Suppose that D is the domain of the model.

A **substitution** is any function $s : \mathcal{V} \to D$ ($\mathcal{V}$ the set of variables).

A substitution s' is said to be an x-**variant** of s, denoted $s \sim_x s'$, if for all $y \in \mathcal{V}$, if $y \neq x$, then $s(y) = s'(y)$.

First Order Interpretations

Let D be the domain.

An **interpretation** I assigns an n-ary relation to each n-ary predicate symbol and an element of the domain to each constant symbol:

If P is an n-ary predicate symbol, then $I(P) \subseteq D^n$

Interpretation in a Kripke Model

Let D be the domain for a Kripke model with worlds W.

An **interpretation** I assigns an n-ary relation to each n-ary predicate symbol and world w and an element of the domain to each constant symbol and world w:

If P is an n-ary predicate symbol, then $I(P, w) \subseteq D^n$

Truth

Let $\mathcal{M} = \langle W, R, D, I \rangle$ be a (varying/constant) domain Kripke model:

- $\mathcal{M}, w \models_s x = y$ iff $s(x) = s(y)$
- $\mathcal{M}, w \models_s P(x_1, \ldots, x_n)$ iff $\langle s(x_1), \ldots, s(x_n) \rangle \in I(P, w)$
- $\mathcal{M}, w \models_s \neg\varphi$ iff $\mathcal{M}, w \not\models_s \varphi$
- $\mathcal{M}, w \models_s \varphi \wedge \psi$ iff $\mathcal{M}, w \models_s \varphi$ and $\mathcal{M}, w \models_s \psi$

Varying Domains

Let $\mathcal{M} = \langle W, R, \mathcal{D}, I \rangle$ be a varying domain Kripke model:

- $\mathcal{M}, w \models_s \Box\varphi$ iff for all $v \in W$, if wRv, then $\mathcal{M}, v \models_s \varphi$
- $\mathcal{M}, w \models_s \forall x\varphi$ iff for all s', if $s \sim_x s'$ and $s'(x) \in \mathcal{D}(w)$, then $\mathcal{M}, w \models_{s'} \varphi$

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- *Actualist* quantification: only quantifying over objects that exist
- $\forall x P(x) \to P(y)$ is *not* valid (cf. *Free logic*)
- Can add possibilist quantifiers
- We can say "y exists": $\exists x(x = y)$,
  "y doesn't exists": $\neg\exists x(x = y)$,
  but we cannot express "there are non-existents"

Constant Domain Models

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- *Possibilist* quantification: quantifying over all objects (even non-existent objects)
- $\forall x P(x) \to P(y)$ is valid
- Can add actualist quantifiers:
  - Introduce an **existence predicate** E (typically assume $I(E, w) \neq \emptyset$ for all $w \in W$ and $\bigcup_w I(E, w) = D$)
  - $\forall^E x\varphi \ := \ \forall x(E(x) \to \varphi)$
  - $\exists^E x\varphi \ := \ \exists x(E(x) \wedge \varphi)$

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Lemma. CBF is valid in a varying domain relational frame iff the frame is monotonic.

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Lemma. BF is valid in a varying domain relational frame iff the frame is anti-monotonic

A varying domain is **anti-monotonic** if for all $w, v \in W$, if wRv, then $\mathcal{D}(v) \subseteq \mathcal{D}(w)$

- Since varying domain semantics can be simulated using constant domain semantics and relativized quantifiers, from a semantic point of view there is really little point in studying the varying domain version in much detail.

- Axiomatic systems intended for constant domain systems have more complex completeness proofs.

Rigidity

Predicates and constants are *not* **rigid**. Their interpretation changes from world to world.

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- $(x = y) \to \Box(x = y)$ is valid
- $(x \neq y) \to \Box(x \neq y)$ is valid
- How should we interpret $\Diamond P(c)$, where c is a **constant**? Two possibilities:
  - The current interpretation of c has the "Possible-P" property
  - there is a possible world such that c (interpreted in that possible world) has the property P

M. Fitting. *Intensional Logic*. Stanford Encyclopedia of Philosophy, 2006. Substantive revision 2015.

M. Fitting. *First-order intensional logic*. Annals of Pure and Applied Logic, 127: 171–193, 2004.

First Order Intensional Logic

In addition to objects there will be what we call *intensions* or *intensional objects* or *concepts*.

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Typical informal intensions are *the morning star*, *the oldest person in the world*, or simply *that*.

Intensions designate different objects under different circumstances—they are non-rigid designators.

They will be modeled by functions from possible worlds to objects. There will be quantification over intensions, as well as quantification over objects.

An intension f picks out an object at each world.

Given a unary predicate P, $P(f)$ could mean the intension f has the property P or the object designated by f has the property P. (Both make sense.)

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De Re/De Dicto issues:

- $P(f)$ is true at w if the object picked out by f at w has property P
- What about $\Diamond P(f)$?
  - (de re) $\Diamond P(f)$ is true at w if the object picked out by f at w has the property P at an accessible world v
  - (de dicto) $\Diamond P(f)$ is true at w if there is an accessible world v such that the object picked out by f at v has the property P

Predicate Abstraction

- (de re) $\Diamond P(f)$ is true at w if the object picked out by f at w has the property P at an accessible world v.

  $\langle \lambda x.\Diamond P(x)\rangle(f)$

- (de dicto) $\Diamond P(f)$ is true at w if there is an accessible world v such that the object picked out by f at v has the property P.

  $\Diamond\langle \lambda x.P(x)\rangle(f)$

Suppose that the possible worlds are people, and f is the *favorite-book* concept picking out, for each person, that person's favorite book. And suppose P is intended to be the *is-an-important-concept* predicate.

For a person who considers reading important, P(f) will most likely be true—the concept of a favorite book would be important for that person.

Let us say Q is intended to be the *is-an-important-book* predicate.

I certainly think $\langle \lambda x.Q(x)\rangle(f)$ is true—for me it says my favorite book is an important book (for me).

I would not think $\langle \lambda x.\Box Q(x)\rangle(f)$ to be true—for me it says that my favorite book is an important book for everybody.

On the other hand I probably would think that $\Box\langle \lambda x.Q(x)\rangle(f)$ is true—for me it says that everybody thinks their favorite book is important.

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$\langle \lambda y. \Diamond \langle \lambda x. T(x, y) \rangle (m) \rangle (m)$

A FOIL model is a structure $\mathcal{M} = \langle W, R, D_O, D_I, I \rangle$, where $W \neq \emptyset$, $R \subseteq W \times W$, D_O is a non-empty set of objects, and D_I is a non-empty set of functions from W to D_O. Finally, I is an interpretation assigning to each predicate symbol P a relation of an appropriate type.

$\mathcal{M}, w \models_s \langle \lambda x.\varphi \rangle (f)$ iff $\mathcal{M}, w \models_{s'} \varphi$ where for all $y \in \mathcal{V}$, if $y \neq x$, then $s'(y) = s(y)$ and $s'(x) = s(f)(w)$.

Valid:

$$\forall x \forall y ((x = y) \to \Box(x = y)))$$

$$\forall x \forall y ((x \neq y) \to \Box(x \neq y)))$$

$$\forall f \forall g [\langle \lambda x, y.(x = y) \rangle (f, g) \to \langle \lambda x, y.\Box(x = y) \rangle (f, g)]$$

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Not Valid:

$$\forall f \forall g[\langle \lambda x, y.(x = y)\rangle(f, g) \to \Box\langle \lambda x, y.(x = y)\rangle(f, g)]$$

Constants and Function Symbols

M. Fitting. *On Height and Happiness*. in Rohit Parikh on Logic, Language and Society, Springer Outstanding Contributions to Logic, C. Baskent, L. Moss, R. Ramanujam editors, pages 235-258, 2017.

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Problem: Names are rigid.

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So, the above two formulas imply:

$\langle \lambda x. \Diamond T(x, x) \rangle (a)$

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