First Order Modal Logic

Eric Pacuit, University of Maryland

November 29, 2023

From Propositional to First Order Modal Logic

“[W]hat is first-order modal logic for? What do quantifiers add to the mix?

From Propositional to First Order Modal Logic

"[W]hat is first-order modal logic for? What do quantifiers add to the mix? Motivations based on natural language and philosophy are still central, though we have a much richer variety of things we can potentially formalize and investigate. Of course we want a semantics that agrees with our intuitive understanding, but now intuitions can, and do, differ substantially from person to person.

From Propositional to First Order Modal Logic

"[W]hat is first-order modal logic for? What do quantifiers add to the mix? Motivations based on natural language and philosophy are still central, though we have a much richer variety of things we can potentially formalize and investigate. Of course we want a semantics that agrees with our intuitive understanding, but now intuitions can, and do, differ substantially from person to person. Are designators rigid? Can objects exist in more than one possible world? Should there be a distinction between identity and necessary identity? And for that matter, is the whole subject a mistake from the beginning, as Quine would have it? Rather than a semantics on which we all generally agree, quite a disparate range has been proposed. We are still exploring what first-order modal semantics should be; the propositional case was settled long ago."

(Fitting, pg. 1, First Order Intensional Logic)

Overviews

M. Fitting and R. Mendelsohn. *First-Order Modal Logic*. Springer, 2nd Edition (2023).

T. Braüner and S. Ghilardi. *First-order Modal Logic*. Handbook of Modal Logic, pgs. 549 - 620 (2007).

D. Gabbay, V. Shehtman and D. Skvortsov. *Quantification in Nonclassical Logic*. Elsevier, 2009.

First Order Modal Language

Let $\mathcal{V}$ be a set of variables.

A **term** is any variable: $\mathcal{T} = \mathcal{V}$ (we will extend the set of terms later when we add **constants** and **function symbols** to the language)

First Order Modal Language

Let $\mathcal{V}$ be a set of variables.

A **term** is any variable: $\mathcal{T} = \mathcal{V}$ (we will extend the set of terms later when we add **constants** and **function symbols** to the language)

Let *Pred* be a set of predicate symbols.

First Order Modal Language

Let $\mathcal{V}$ be a set of variables.

A **term** is any variable: $\mathcal{T} = \mathcal{V}$ (we will extend the set of terms later when we add **constants** and **function symbols** to the language)

Let *Pred* be a set of predicate symbols.

A **formula** is constructed as follows

$$\varphi \quad := \quad t_1 = t_2 \mid P(t_1, \ldots, t_n) \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid \Diamond\varphi \mid (\forall x)\varphi \mid (\exists x)\varphi$$

where $P \in \mathit{Pred}$ of arity n, $t_i \in \mathcal{T}$ for $i = 1, \ldots, n$ and $x \in \mathcal{V}$

(Sometimes equality is not in the language)

- $\forall x \Box P(x)$ $\exists x \Diamond P(x)$
- $\Box \forall x P(x)$ $\Diamond \exists x P(x)$

- $\forall x \Box P(x)$ $\exists x \Diamond P(x)$
- $\Box \forall x P(x)$ $\Diamond \exists x P(x)$

- $\exists x \Box P(x)$
- $\Box \exists x P(x)$

- $\forall x \Box P(x)$ $\exists x \Diamond P(x)$
- $\Box \forall x P(x)$ $\Diamond \exists x P(x)$

- $\exists x \Box P(x)$
- $\Box \exists x P(x)$

- $\Box \forall x \varphi(x) \to \forall x \Box \varphi(x)$
- $\forall x \Box \varphi(x) \to \Box \forall x \varphi(x)$

De Re/De Dicto Ambiguity

Everything is necessarily F

De Re/De Dicto Ambiguity

Everything is necessarily F

de dicto: It is necessary that everything is F

de re: Each thing is such that it is necessarily F

De Re/De Dicto Ambiguity

Everything is necessarily F

de dicto: It is necessary that everything is F

$\Box(\forall x)F(x)$

de re: Each thing is such that it is necessarily F

$(\forall x)\Box F(x)$

De Re/De Dicto Ambiguity

Something necessarily exists

de dicto: It is necessary that something exists

de re: There is something that exists necessarily

De Re/De Dicto Ambiguity

Something necessarily exists

de dicto: It is necessary that something exists

$\Box(\exists x)E(x)$

de re: There is something that exists necessarily

$(\exists x)\Box E(x)$

De Re/De Dicto Ambiguity

The number of planets is necessarily even

De Re/De Dicto Ambiguity

The number of planets is necessarily even

de dicto: The proposition *that the number of planets is even* is necessary.

de re: Of the number of planets, that number is necessarily even.

De Re/De Dicto Ambiguity

The number of planets is necessarily even

de dicto: The proposition *that the number of planets is even* is necessary.

$\Box\langle\lambda x.x \text{ is even}\rangle(t)$

de re: Of the number of planets, that number is necessarily even.

$\langle\lambda x.\Box(x \text{ is even})\rangle(t)$

Lambda Notation

Describing Functions:

- $f : \mathbb{R} \to \mathbb{R}$, where for all $x \in \mathbb{R}$, $f(x) = x^2$
- $x \mapsto x^2$
- $\lambda x.x^2$

Beta Reduction:

- $f(3) = 9$
- $(\lambda x.x^2)(3) = 3^2$

De Re/De Dicto Ambiguity

There is no de re/de dicto ambiguity in formulas with free variables: E.g.,
$\Box(x \text{ is a Democrat})$

De Re/De Dicto Ambiguity

There is no de re/de dicto ambiguity in formulas with free variables: E.g.,
$\Box(x \text{ is a Democrat})$

The U.S. President will always be a Democrat.
Understood *de re* this is true, but understood *de dicto*, this is not true.

Joe Biden will always be a Democrat.
Understood both *de re* and *de dicto*, this is true.

De Re/De Dicto Ambiguity

The U.S. President will always be a Democrat.
Understood *de re* this is true, but understood *de dicto*, this is not true.

Joe Biden will always be a Democrat.
Understood both *de re* and *de dicto*, this is true.

> Does the difference in truth value show that *temporality* has more to do with how the object is specified than with the object itself? Hardly. It depends on the fact that the Presidency will be changing hands, and Joe Biden only temporarily holds that office. In the world of 2022, the two coincide; but in later worlds, then don't.
>
> (Fitting & Mendelsohn, p. 170)

Constant vs. Varying Domains

A **Kripke frame** is a tuple $\langle W, R \rangle$ where $W \neq \emptyset$ and $R \subseteq W \times W$ is a relation on W

Constant vs. Varying Domains

A **Kripke frame** is a tuple $\langle W, R \rangle$ where $W \neq \emptyset$ and $R \subseteq W \times W$ is a relation on W

A **constant domain Kripke skeleton** is a tuple $\langle W, R, D \rangle$ where $\langle W, R \rangle$ is a frame and $D \neq \emptyset$ is a non-empty set.

Constant vs. Varying Domains

A **Kripke frame** is a tuple $\langle W, R \rangle$ where $W \neq \emptyset$ and $R \subseteq W \times W$ is a relation on W

A **constant domain Kripke skeleton** is a tuple $\langle W, R, D \rangle$ where $\langle W, R \rangle$ is a frame and $D \neq \emptyset$ is a non-empty set.

A **varying domain Kripke skeleton** is a tuple $\langle W, R, \mathcal{D} \rangle$ where $\langle W, R \rangle$ is a frame and for each $w \in W$, $\mathcal{D}(w)$ is a set (the domain at w). Let the **domain of the model** be $D = \bigcup_{w \in W} \mathcal{D}(w)$.

Substitutions

Suppose that D is the domain of the model.

A **substitution** is any function $s : \mathcal{V} \to D$ ($\mathcal{V}$ the set of variables).

A substitution s' is said to be an x-**variant** of s, denoted $s \sim_x s'$, if for all $y \in \mathcal{V}$, if $y \neq x$, then $s(y) = s'(y)$.

First Order Interpretations

Let D be the domain.

An **interpretation** I assigns an n-ary relation to each n-ary predicate symbol and an element of the domain to each constant symbol:

If P is an n-ary predicate symbol, then $I(P) \subseteq D^n$

If $t \in \mathcal{T}$ is a term, I is an interpretation and s is a substitution, then $t^{I,s} \in D$, where $t^{I,s}$ is $s(t)$ if $t \in \mathcal{V}$

Interpretation in a Kripke Model

Let D be the domain for a Kripke model with worlds W.

An **interpretation** I assigns an n-ary relation to each n-ary predicate symbol and world w and an element of the domain to each constant symbol and world w:

If P is an n-ary predicate symbol, then $I(P, w) \subseteq D^n$

If $t \in \mathcal{T}$ is a term, I is an interpretation and s is a substitution and $w \in W$, then $t^{I,s,w} \in D$, where $t^{I,s,w}$ is $s(t)$ if $t \in \mathcal{V}$

Truth

Let $\mathcal{M} = \langle W, R, D, I \rangle$ be a (varying/constant) domain Kripke model:

- $\mathcal{M}, w \models_s t_1 = t_2$ iff $t_1^{I,s,w} = t_2^{I,s,w}$
- $\mathcal{M}, w \models_s P(t_1, \ldots, t_n)$ iff $\langle t_1^{I,s,w}, \ldots, t_n^{I,s,w} \rangle \in I(P, w)$
- $\mathcal{M}, w \models_s \neg\varphi$ iff $\mathcal{M}, w \not\models_s \varphi$
- $\mathcal{M}, w \models_s \varphi \wedge \psi$ iff $\mathcal{M}, w \models_s \varphi$ and $\mathcal{M}, w \models_s \psi$

Varying Domains

Let $\mathcal{M} = \langle W, R, \mathcal{D}, I \rangle$ be a varying domain Kripke model:

- $\mathcal{M}, w \models_s \Box\varphi$ iff for all $v \in W$, if wRv, then $\mathcal{M}, v \models_s \varphi$
- $\mathcal{M}, w \models_s \forall x\varphi$ iff for all s', if $s \sim_x s'$ and $s'(x) \in \mathcal{D}(w)$, then $\mathcal{M}, w \models_{s'} \varphi$

Varying Domains

Let $\mathcal{M} = \langle W, R, \mathcal{D}, I \rangle$ be a varying domain Kripke model:

- $\mathcal{M}, w \models_s \Box\varphi$ iff for all $v \in W$, if wRv, then $\mathcal{M}, v \models_s \varphi$
- $\mathcal{M}, w \models_s \forall x\varphi$ iff for all s', if $s \sim_x s'$ and $s'(x) \in \mathcal{D}(w)$, then $\mathcal{M}, w \models_{s'} \varphi$

- *Actualist* quantification: only quantifying over objects that exist
- $\forall x P(x) \to P(y)$ is *not* valid (cf. *Free logic*)
- Can add possibilist quantifiers

Varying Domains

Let $\mathcal{M} = \langle W, R, \mathcal{D}, I \rangle$ be a varying domain Kripke model:

- $\mathcal{M}, w \models_s \Box\varphi$ iff for all $v \in W$, if wRv, then $\mathcal{M}, v \models_s \varphi$
- $\mathcal{M}, w \models_s \forall x\varphi$ iff for all s', if $s \sim_x s'$ and $s'(x) \in \mathcal{D}(w)$, then $\mathcal{M}, w \models_{s'} \varphi$

- *Actualist* quantification: only quantifying over objects that exist
- $\forall x P(x) \to P(y)$ is *not* valid (cf. *Free logic*)
- Can add possibilist quantifiers
- We can say "y exists": $\exists x(x = y)$,
  "y doesn't exists": $\neg\exists x(x = y)$,
  but we cannot express "there are non-existents"

Constant Domain Models

Let $\mathcal{M} = \langle W, R, D, I \rangle$ be a constant domain Kripke model:

- $\mathcal{M}, w \models_s \Box\varphi$ iff for all $v \in W$, if wRv, then $\mathcal{M}, v \models_s \varphi$
- $\mathcal{M}, w \models_s \forall x\varphi$ iff for all s', if $s \sim_x s'$, then $\mathcal{M}, w \models_{s'} \varphi$

Constant Domain Models

Let $\mathcal{M} = \langle W, R, D, I \rangle$ be a constant domain Kripke model:

- $\mathcal{M}, w \models_s \Box\varphi$ iff for all $v \in W$, if wRv, then $\mathcal{M}, v \models_s \varphi$
- $\mathcal{M}, w \models_s \forall x\varphi$ iff for all s', if $s \sim_x s'$, then $\mathcal{M}, w \models_{s'} \varphi$

- *Possibilist* quantification: quantifying over all objects (even non-existent objects)
- $\forall x P(x) \to P(y)$ is valid
- Can add actualist quantifiers:
  - Introduce an **existence predicate** E (typically assume $I(E, w) \neq \emptyset$ for all $w \in W$ and $\bigcup_w I(E, w) = D$)
  - $\forall^E x\varphi \ := \ \forall x(E(x) \to \varphi)$
  - $\exists^E x\varphi \ := \ \exists x(E(x) \wedge \varphi)$

Barcan Schemas

Barcan Schemas

- **Barcan formula** (BF): $\forall x \Box \varphi(x) \to \Box \forall x \varphi(x)$
- **converse Barcan formula** (CBF): $\Box \forall x \varphi(x) \to \forall x \Box \varphi(x)$

Barcan Schemas

- **Barcan formula** (BF): $\forall x \Box \varphi(x) \to \Box \forall x \varphi(x)$
- **converse Barcan formula** (CBF): $\Box \forall x \varphi(x) \to \forall x \Box \varphi(x)$

Lemma. CBF is valid in a varying domain relational frame iff the frame is monotonic.

A varying domain is **monotonic** if for all $w, v \in W$, if wRv, then $\mathcal{D}(w) \subseteq \mathcal{D}(v)$

Barcan Schemas

- **Barcan formula** (BF): $\forall x \Box \varphi(x) \rightarrow \Box \forall x \varphi(x)$
- **converse Barcan formula** (CBF): $\Box \forall x \varphi(x) \rightarrow \forall x \Box \varphi(x)$

Lemma. CBF is valid in a varying domain relational frame iff the frame is monotonic.

A varying domain is **monotonic** if for all $w, v \in W$, if wRv, then $\mathcal{D}(w) \subseteq \mathcal{D}(v)$

Lemma. BF is valid in a varying domain relational frame iff the frame is anti-monotonic

A varying domain is **anti-monotonic** if for all $w, v \in W$, if wRv, then $\mathcal{D}(v) \subseteq \mathcal{D}(w)$