Modal Logic:
Incompleteness and Non-Normal Modal Logics

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Semantics for Modal Logic

- Relational Models
- Neighborhood Models
- Algebraic Models
- Models based on General Frames

A logic $\mathsf{L} \subseteq \mathcal{L}$ is a **normal modal logic** if

- L contains all tautologies of classical propositional logic
- L is closed under modus ponens
- L is closed under uniform substitution
- L is closed under necessitation
- $\Box(p \to q) \to (\Box p \to \Box q) \in \mathsf{L}$

Let K be the smallest normal modal logic.

φ is **globally true** in a Kripke model $\mathcal{M}$, written $\mathcal{M} \models \varphi$, if $\mathcal{M}, w \models \varphi$ for all $w \in \mathcal{M}$

φ is **valid** in a Kripke frame $\mathcal{F}$, written $\mathcal{F} \models \varphi$, if $\mathcal{M} \models \varphi$ for all $\mathcal{M}$ based on $\mathcal{F}$

φ is **valid over a class $\mathbb{F}$ of frames** if for all $\mathcal{F} \in \mathbb{F}$, $\mathcal{F} \models \varphi$

For a class $\mathbb{F}$ of frames, let $Log(\mathbb{F}) = \{\varphi \mid \mathcal{F} \models \varphi \text{ for all } \mathcal{F} \in \mathbb{F}\}$

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Let $\mathit{Fr}(\mathbf{L}) = \{\mathcal{F} \mid \mathcal{F} \models \varphi \text{ for all } \varphi \in \mathbf{L}\}$

Theorem (Thomason 1972; Fine 1975, Thomason 1974). There are Kripke incomplete logics.

Warm-up exercises

- $\mathcal{F} \models (\Diamond\varphi \wedge \Diamond\psi) \rightarrow (\Diamond(\varphi \wedge \Diamond\psi) \vee \Diamond(\varphi \wedge \psi) \vee \Diamond(\Diamond\varphi \wedge \psi))$ iff $\mathcal{F}$ non-branching to the right (for all w, v, x if wRv and wRx then either vRx or xRv or $v = x$).

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- $\mathcal{F} \models \Box\varphi \rightarrow \Diamond\varphi$ iff $\mathcal{F}$ is unbounded to the right (for all w there is a v such that wRv).

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- $\mathcal{F} \models \Box(\Box\varphi \to \varphi) \to \Box\varphi$ iff $\mathcal{F}$ is transitive and converse well-founded.

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  $\mathcal{M}, 0 \models \Box\Diamond p$: For every number greater than 0, there is some larger number that is even.
  $\mathcal{M}, 0 \not\models \Diamond\Box p$: There is no number greater than 0 such that every larger number is even.

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- For any model $\mathcal{M} = \langle \mathbb{N}, <, V \rangle$, if $V(p)$ is finite or cofinite, then for all $n \in \mathbb{N}$

$$\mathcal{M}, n \models \Box\Diamond p \to \Diamond\Box p$$

Temporal Logic

Let $\mathcal{M} = \langle T, R, V \rangle$ be a Kripke model.

- $\mathcal{M}, t \models F\varphi$ iff there exists a t' such that tRt' and $\mathcal{M}, t' \models \varphi$
- $\mathcal{M}, t \models P\varphi$ iff there exists a t' such that $t'Rt$ and $\mathcal{M}, t' \models \varphi$
- $\mathcal{M}, t \models G\varphi$ iff for all t', if tRt' then $\mathcal{M}, t' \models \varphi$
- $\mathcal{M}, t \models H\varphi$ iff for all t', if $t'Rt$ then $\mathcal{M}, t' \models \varphi$

Temporal Logic

The minimal temporal logic $\mathbf{K_t}$ contains the following axiom schemes and rules:

- Propositional logic
- $G(\varphi \to \psi) \to (G\varphi \to G\psi)$
- $H(\varphi \to \psi) \to (H\varphi \to H\psi)$
- $\varphi \to GP\varphi$
- $\varphi \to HF\varphi$
- From φ infer $G\varphi$
- From φ infer $H\varphi$
- Modus Ponens

Temporal Logic

Let $\mathbf{K_t Tho}$ be the temporal logic extending $\mathbf{K_t}$ with the axiom schemes

- $Fp \wedge Fq \to (F(p \wedge Fq) \vee F(p \wedge q) \vee F(Fp \wedge q))$
- $Gp \to Fp$
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$\mathbf{K_t ThoM}$ extends $\mathbf{K_t Tho}$ with the axiom scheme $GF\varphi \to FG\varphi$.

Fact

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The logic $\mathbf{K_t ThoM}$ Kripke incomplete *(i.e.,* $\mathbf{K_t ThoM}$ *is not the logic of any class of frames).*

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- Boolean Algebra with Operators
- General Frames

Lattice

A **lattice** is an algebra $\mathcal{A} = (A, \wedge, \vee)$ where A is a set (called the *carrier set* or the *domain*) and $\wedge$ and $\vee$ are binary operators (i.e., functions mapping pairs of elements from A to elements of A) satisfying the following equations: for all $x, y, z \in A$:

(1a)	$x \vee x = x$	(1b)	$x \wedge x = x$
(2a)	$x \vee y = y \vee x$	(2b)	$x \wedge y = y \wedge x$
(3a)	$x \vee (y \vee z) = (x \vee y) \vee z$	(2b)	$x \wedge (y \wedge z) = (x \wedge y) \wedge z$
(4a)	$x \vee (x \wedge y) = x$	(4b)	$x \wedge (x \vee y) = x$

Boolean Algebra

$\mathcal{A} = (A, \wedge, \vee)$ is a **distributive lattice** if $\mathcal{A}$ is a lattice and the following equations are satisfied: for all $x, y, z \in A$

(5a) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ (5b) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

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A (distributive) lattice $\mathcal{A}$ is **bounded** if there are $0 \in A$ and $1 \in A$ such that: for all $x \in A$,

$$\begin{array}{llll} (6a) & x \vee 1 = 1 & (6b) & x \wedge 1 = x \\ (7a) & x \vee 0 = x & (7b) & x \wedge 0 = 0 \end{array}$$

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The structure $\mathcal{A} = (A, \wedge, \vee, -)$ is **Boolean algebra** if $(A, \wedge, \vee)$ is a bounded distributive lattice, $-$ is a unary operator on A satisfying the following equations: for all $x \in A$,

$$(8a) \quad x \vee -x = 1 \qquad (8b) \quad x \wedge -x = 0$$

Examples of Boolean Algebras

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- Suppose that $\mathcal{S} \subseteq \wp(W)$ is closed under $\cap$, $\cup$ and $\bar{\cdot}$. Then $(\mathcal{S}, \cap, \cup, \varnothing, W)$ is a Boolean algebra. It is a **subalgebra** of $\mathbf{2}^W$.

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- Let $\mathcal{S} = \{X \subseteq \mathbb{N} \mid X \text{ is finite or } \mathbb{N} \setminus X \text{ is finite}\}$. Then $(\mathcal{S}, \cup, \cap, -, \varnothing, \mathbb{N})$ is a Boolean algebra.

Boolean Algebra with Operators

A *BAO* is a **Boolean algebra together with one more more unary operators** f such that $f(x \vee y) = f(x) \vee f(y)$ and for the bottom element of the algebra 0, $f(0) = 0$.

We often denote the operator f by '$\Diamond$'. So, a BAO is a tuple $\langle A, \wedge, \vee, \neg, 0, 1, \Diamond \rangle$ where A is a set and all the axioms 1a-8a, 1b-8b are all satisfied and $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$ and $\Diamond 0 = 0$.

General Frames

General frames/models: $\langle W, R, \mathcal{A} \rangle$ where $\langle W, R \rangle$ is a frame, and $\mathcal{A} \subseteq \wp(W)$ is a BAO: Boolean algebra closed under the operator $R^{-1} : \wp(W) \to \wp(W)$, where for all X,

$$R^{-1}(X) = \{w \mid \text{there is a } v \in X \text{ with } w \mathrel{R} v\}.$$

A general model is a structure $\langle W, R, \mathcal{A}, V \rangle$, where $\langle W, R, \mathcal{A} \rangle$ is a general frame and for all $p \in \mathrm{At}$, $V(p) \in \mathcal{A}$.

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A general model is a structure $\langle W, R, \mathcal{A}, V \rangle$, where $\langle W, R, \mathcal{A} \rangle$ is a general frame and for all $p \in \text{At}$, $V(p) \in \mathcal{A}$.

Theorem. Every consistent modal logic is sound and complete with respect to some class of general frames.

Fact

The logic $\mathbf{K_t ThoM}$ *is consistent.*

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Consider the general frame $\mathcal{F} = \langle \mathbb{N}, <, \mathcal{A} \rangle$, where

$$\mathcal{A} = \{X \mid X \subseteq \mathbb{N} \text{ and } X \text{ is finite or cofinite}\}.$$

We have the following:

- $\mathcal{F} \models GF\varphi \to FG\varphi$, since for every general model based on this frame, the set of states that make φ true is finite or cofinite
- $\mathcal{F}$ validates the axioms of $\mathbf{K_t Tho}$ since the underlying frame $\langle \mathbb{N}, < \rangle$ validates the axioms.
- $\mathcal{F}$ validates $\mathbf{K_t ThoM}$

$$(vB) \qquad \Box\Diamond\top \to \Box(\Box(\Box p \to p) \to p)$$

Let vB be the smallest normal modal logic containing vB.

Theorem (van Benthem, 1979)

The logic vB *is incomplete.*

Lemma

Any Kripke frame that validates vB *also validates* $\Box\Diamond\top \to \Box\bot$.

Lemma

$\Box\Diamond\top \to \Box(\Box(\Box p \to p) \to p)$ is valid over $\mathcal{VB}$ while $\Box\Diamond\top \to \Box\bot$ is not. Thus, $\Box\Diamond\top \to \Box\bot \notin \mathsf{vB}$.

Definition (van Benthem Frame)

Let $\mathcal{VB} = \langle W, R, \mathbb{W} \rangle$ where:

1. $W = \mathbb{N} \cup \{\infty, \infty + 1\}$;
2. $R = \{(\infty + 1, \infty), (\infty, \infty)\} \cup \{(\infty, n) \mid n \in \mathbb{N}\} \cup \{(m, n) \mid m, n \in \mathbb{N}, m > n\}$;
3. $\mathbb{W} = \{X \subseteq W \mid X \text{ is finite and } \infty \notin X \} \cup \{X \subseteq W \mid X \text{ is cofinite and } \infty \in X \}$

Lemma

$\Box\Diamond\top \to \Box(\Box(\Box p \to p) \to p)$ is valid over $\mathcal{VB}$ while $\Box\Diamond\top \to \Box\bot$ is not.

Incompleteness for Neighborhood Frames

Are all modal logics complete with respect to some class of neighborhood frames?

Incompleteness for Neighborhood Frames

Are all modal logics complete with respect to some class of neighborhood frames? No

Incompleteness

Martin Gerson. *The Inadequacy of Neighbourhood Semantics for Modal Logic*. Journal of Symbolic Logic (1975).

There are two logics $\mathbf{L}$ and $\mathbf{L}'$ that are incomplete with respect to neighborhood semantics.

Incompleteness

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There are two logics **L** and **L**$'$ that are incomplete with respect to neighborhood semantics.

L is between **T** and **S4**

L$'$ is above **S4** (adapts Fine's incomplete logic)

Extra Slides

Comparing Relational and Neighborhood Semantics

Fact: If a (normal) modal logic is complete with respect to some class of relational frames then it is complete with respect to some class of neighborhood frames.

What about the converse?

Are there normal modal logics that are incomplete with respect to relational semantics, but complete with respect to neighborhood semantics?

Comparing Relational and Neighborhood Semantics

Fact: If a (normal) modal logic is complete with respect to some class of relational frames then it is complete with respect to some class of neighborhood frames.

What about the converse?

Are there normal modal logics that are incomplete with respect to relational semantics, but complete with respect to neighborhood semantics? Yes!

Comparing Relational and Neighborhood Semantics

Neighborhood completeness does not imply Kripke completeness

- extension of **K**

D. Gabbay. *A normal logic that is complete for neighborhood frames but not for Kripke frames*. Theoria (1975).

- extension of **T**

M. Gerson. *A Neighbourhood frame for T with no equivalent relational frame*. Zeitschr. J. Math. Logik und Grundlagen (1976).

- extension of **S4**

M. Gerson. *An Extension of S4 Complete for the Neighbourhood Semantics but Incomplete for the Relational Semantics*. Studia Logica (1975).

W. Holliday and T. Litak. *Complete Additivity and Modal Incompleteness*. The Review of Symbolic Logic, 2020.

L. Chagrova. *On the Degree of Neighborhood Incompleteness of Normal Modal Logics*. AiML 1 (1998).

V. Shehtman. *On Strong Neighbourhood Completeness of Modal and Intermediate Propositional Logics (Part I)*. AiML 1 (1998).

T. Litak. *Modal Incompleteness Revisited*. Studia Logica (2004).

A Kripke frame $\mathcal{F} = \langle W, R \rangle$ is associated with its dual $\mathcal{F}^+ = \langle \wp(W), \cap, \cup, -, R^{-1} \rangle$.

Let $\mathfrak{A} = (A, \wedge, \vee, -, \bot, \top, \Diamond)$ be a BAO.

- $\mathcal{C}$: For all $X \subseteq A$, $\bigvee X$ exists and is an element of A
- $\mathcal{A}$: Any non-bottom element is above an **atom**, i.e., minimal non-bottom element (if $a \neq \bot$, then there is a $b \neq \bot$ such that $a > b$ and for all c if $b > c$, then $c = \bot$)
- $\mathcal{V}$: For all $X \subseteq A$, if $\bigvee X$ exists, then

$$\Diamond \bigvee X = \bigvee \{\Diamond x \mid x \in X\}$$

For every Kripke frame $\mathcal{F}$, $\mathcal{F}^+$ is a $\mathcal{CAV}$-*BAO*

Taking any Kripke frame/$\mathcal{CAV}$-BAO, converting it into its dual $\mathcal{CAV}$-BAO/Kripke frame, and then going back produces an output isomorphic to the original input. Therefore, Kripke completeness is just $\mathcal{CAV}$-completeness.

The fact that a normal modal logic is not the logic of any class of Kripke frames means that it is not the logic of any class of $\mathcal{CAV}$-BAO.

Kripke incompleteness is the phenomenon that not every variety of BAOs can be generated as the smallest variety containing some class of $\mathcal{CAV}$-BAOs.

Given that the properties $\mathcal{C}$, $\mathcal{A}$, and $\mathcal{V}$ are independent of each other, will arbitrary combinations of these three lead to distinct notions of completeness, each more general than Kripke completeness but less general than algebraic completeness? Or is the propositional modal language too coarse to care about differences between all or at least some of these semantics?

W. Holliday and Y. Ding. *Another Problem in Possible World Semantics*. Proceedings of AiML, 2020.

Basic modal language: $\varphi \quad := \quad p \mid \neg\varphi \mid (\varphi \wedge \psi) \mid \Box\varphi \mid Q\varphi$
where $p \in \mathsf{At}$

Frame: $\mathcal{M} = \langle W, N_\Box, N_Q \rangle$ where $W \neq \emptyset$, $N_\Box : W \to \wp(\wp(W))$ and $N_Q : W \to \wp(\wp(W))$

Model: $\mathcal{M} = \langle W, N_\Box, N_Q, V \rangle$ where $\langle W, N_\Box, N_Q \rangle$ is a frame and $V : \mathsf{At} \to \wp(W)$

Truth:

- $\mathcal{M}, w \models \Box\varphi$ iff $[\![\varphi]\!]_{\mathcal{M}} \in N_\Box(w)$
- $\mathcal{M}, w \models Q\varphi$ iff $[\![\varphi]\!]_{\mathcal{M}} \in N_Q(w)$

A logic L is congruential if it contains all propositional tautologies, is closed under modus ponens, closed under uniforms substitution and closed under the congruence rule: if $\varphi \leftrightarrow \psi \in \mathsf{L}$, then $O\varphi \leftrightarrow O\psi \in \mathsf{L}$ (for each operator O).

$$(Split) \qquad p \rightarrow (\Diamond(p \wedge Qp) \wedge \Diamond(p \wedge \neg Qp))$$

Let S be the smallest congruential modal logic containing *Split*.

Theorem (Holliday and Ding, 2020) There is no neighborhood frame that validates S; If a *BAO* validates S, then it is atomless; The logic S is complete for a class of neighborhood possibility frames.