Modal Logic: Incompleteness and Non-Normal Modal Logics

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Neighborhood Frames

Let W be a non-empty set of states.

Any function $N : W \to \wp(\wp(W))$ is called a neighborhood function

A pair $\langle W, N \rangle$ is a called a neighborhood frame if W a non-empty set and N is a neighborhood function.

A neighborhood model based on $\mathfrak{F} = \langle W, N \rangle$ is a tuple $\langle W, N, V \rangle$ where $V : \mathrm{At} \to \wp(W)$ is a valuation function.

Truth in a Model

- $\mathfrak{M}, w \models p$ iff $w \in V(p)$
- $\mathfrak{M}, w \models \neg\varphi$ iff $\mathfrak{M}, w \not\models \varphi$
- $\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$

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- $\mathfrak{M}, w \models \Box\varphi$ iff $[\![\varphi]\!]_{\mathfrak{M}} \in N(w)$
- $\mathfrak{M}, w \models \Diamond\varphi$ iff $W - [\![\varphi]\!]_{\mathfrak{M}} \notin N(w)$

where $[\![\varphi]\!]_{\mathfrak{M}} = \{w \mid \mathfrak{M}, w \models \varphi\}$.

Let $N : W \to \wp\wp W$ be a neighborhood function and define $m_N : \wp W \to \wp W$:

$$\text{for } X \subseteq W,\ m_N(X) = \{w \mid X \in N(w)\}$$

1. $[\![p]\!]_{\mathfrak{M}} = V(p)$ for $p \in \mathsf{At}$
2. $[\![\neg\varphi]\!]_{\mathfrak{M}} = W - [\![\varphi]\!]_{\mathfrak{M}}$
3. $[\![\varphi \wedge \psi]\!]_{\mathfrak{M}} = [\![\varphi]\!]_{\mathfrak{M}} \cap [\![\psi]\!]_{\mathfrak{M}}$
4. $[\![\Box\varphi]\!]_{\mathfrak{M}} = m_N([\![\varphi]\!]_{\mathfrak{M}})$
5. $[\![\Diamond\varphi]\!]_{\mathfrak{M}} = W - m_N(W - [\![\varphi]\!]_{\mathfrak{M}})$

Detailed Example

Suppose $W = \{w, s, v\}$ is the set of states and define a neighborhood model $\mathfrak{M} = \langle W, N, V \rangle$ as follows:

- $N(w) = \{\{s\}, \{v\}, \{w, v\}\}$
- $N(s) = \{\{w, v\}, \{w\}, \{w, s\}\}$
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Further suppose that $V(p) = \{w, s\}$ and $V(q) = \{s, v\}$.

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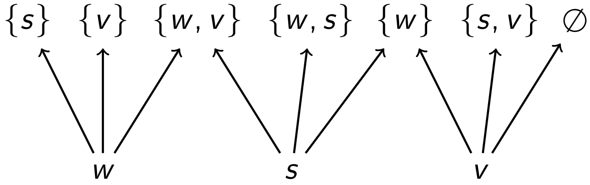

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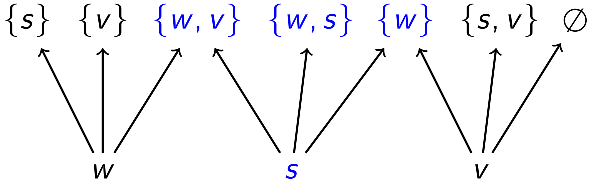

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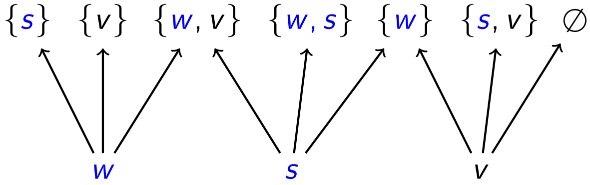

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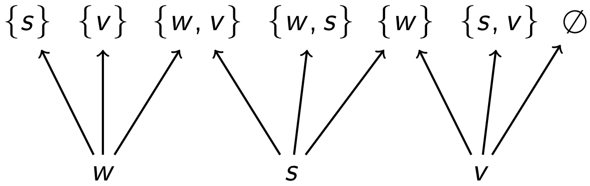

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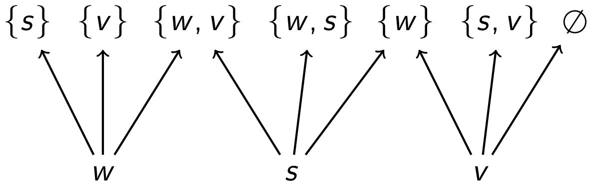

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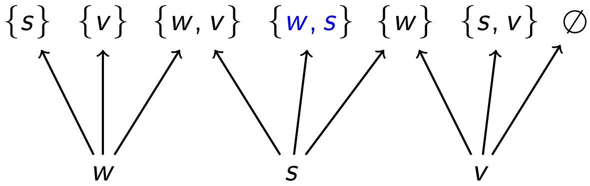

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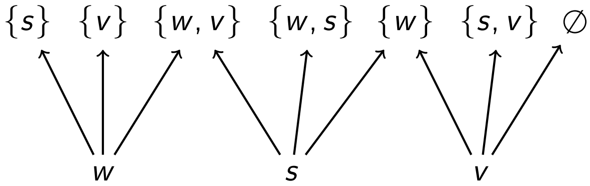

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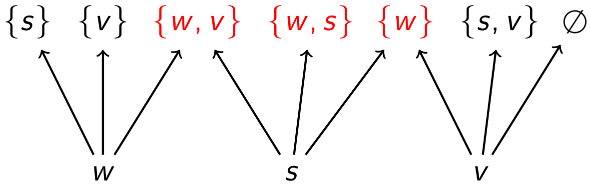

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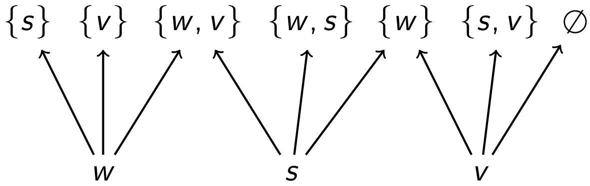

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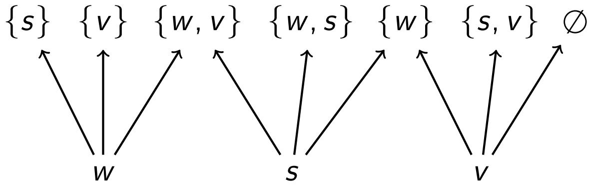

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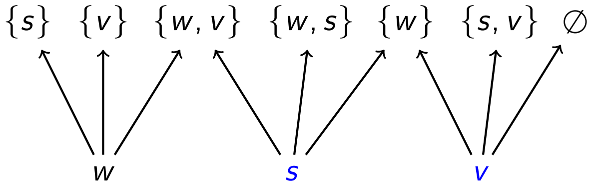

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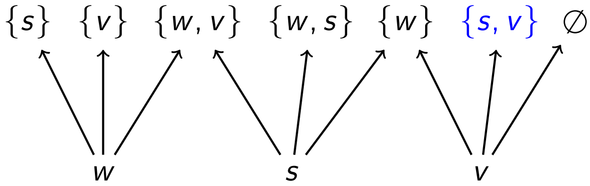

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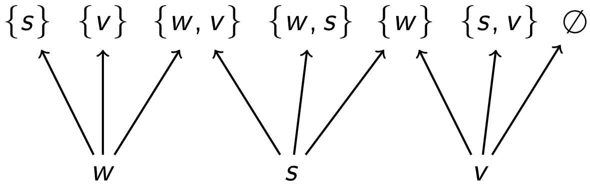

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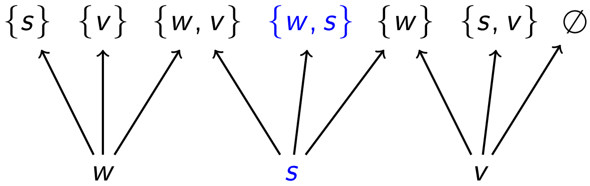

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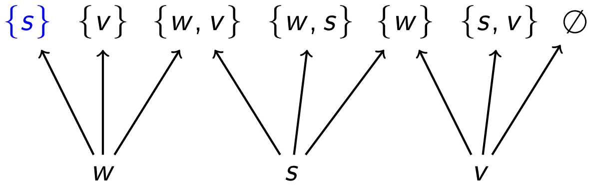

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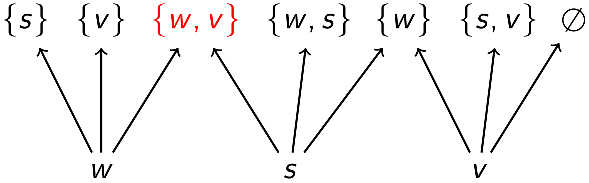

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Defining beliefs from evidence

J. van Benthem and EP. *Dynamic logics of evidence-based beliefs*. Studia Logica, 99(61), 2011.

J. van Benthem, D. Fernández-Duque and EP. *Evidence and plausibility in neighborhood structures*. Annals of Pure and Applied Logic, 165, pp. 106-133.

Evidence Models: Basic Assumptions

Let W be a set of possible worlds or states one of which represents the "actual" situation.

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3. Despite the fact that sources may not be reliable or jointly inconsistent, they are all the agent has for forming beliefs.

Evidential States

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In addition, much of the literature would suggest a 'monotonicity' assumption:

If the agent has evidence X and $X \subseteq Y$ then the agent has evidence Y.

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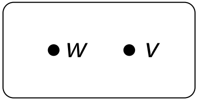
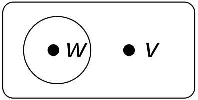
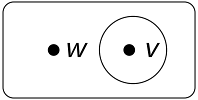


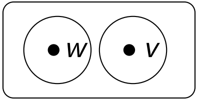
There is no evidence for or against p.



There is evidence that supports p.



There is evidence that rejects p.



There is evidence that supports p and also evidence that rejects p.

Evidence Model

Evidence model: $\mathcal{M} = \langle W, E, V \rangle$

- W is a non-empty set of worlds,
- $V : \mathsf{At} \to \wp(W)$ is a valuation function, and
- $E \subseteq W \times \wp(W)$ is an evidence relation

  $E(w) = \{X \mid w\,E\,X\}$ and $X \in E(w)$: "the agent accepts X as evidence at state w".

Uniform evidence model (E is a constant function): $\langle W, \mathcal{E}, V \rangle, w$ where $\mathcal{E}$ is the fixed family of subsets of W related to each state by E.

Assumptions

(Cons) For each state w, $\emptyset \notin E(w)$.

(Triv) For each state w, $W \in E(w)$.

The Basic Language $\mathcal{L}$ of Evidence and Belief

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi \mid B\varphi \mid A\varphi$$

- $\Box\varphi$: "the agent has evidence that φ is true" (i.e., "the agent has evidence for φ")
- $B\varphi$ says that "the agents believes that φ is true" (based on her evidence)
- $A\varphi$: "φ is true in all states" (for technical convenience/knowledge)

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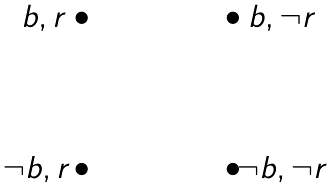

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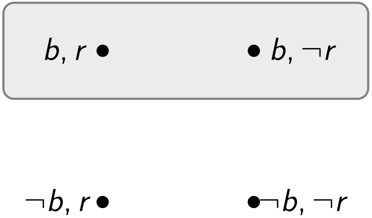

- Receive evidence that the animal is a bird

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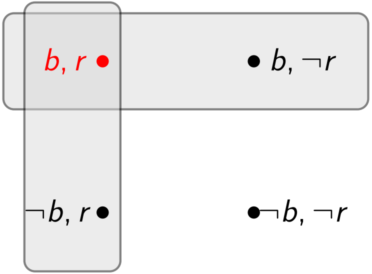

- Receive evidence that the animal is a bird
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- $B(b \wedge r)$

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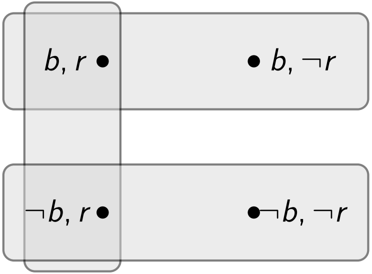

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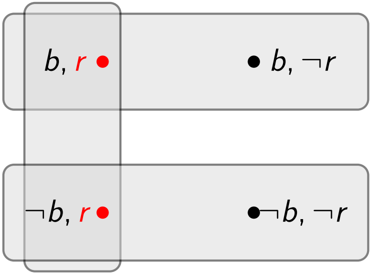

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Defining Beliefs

w-**scenario**: A maximal family of evidence sets $\mathcal{X} \subseteq E(w)$ that has the **finite intersection property** (f.i.p.: for each finite subfamily $\{X_1, \ldots, X_n\} \subseteq \mathcal{X}$, $\bigcap_{1 \leq i \leq n} X_i \neq \emptyset$).

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An agent believes φ at w if each w-scenario implies that φ is true (i.e., φ is true at each point in the intersection of each w-scenario).

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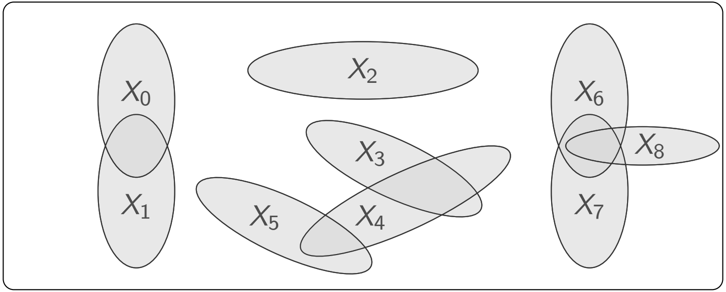

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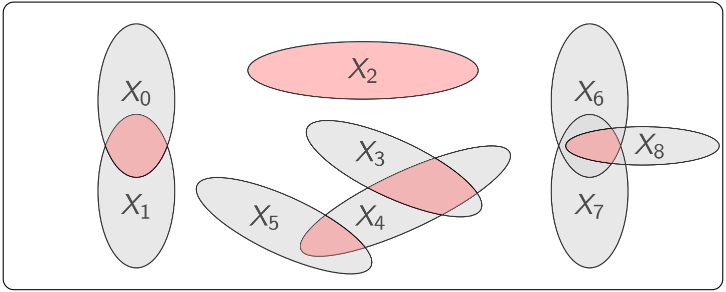

Our definition of belief is very conservative, many other definitions are possible (there exists a w-scenario, "most" of the w-scenarios,...)

Truth

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- $\mathcal{M}, w \models A\varphi$ iff for all $v \in W$, $\mathcal{M}, v \models \varphi$

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- $\mathcal{M}, w \models A\varphi$ iff for all $v \in W$, $\mathcal{M}, v \models \varphi$
- $\mathcal{M}, w \models B\varphi$ for each maximal f.i.p. $\mathcal{X} \subseteq E(w)$ and for all $v \in \bigcap \mathcal{X}$, $\mathcal{M}, v \models \varphi$

Notation for the truth set: $[\![\varphi]\!]_{\mathcal{M}} = \{w \mid \mathcal{M}, w \models \varphi\}$

Flat Evidence Models

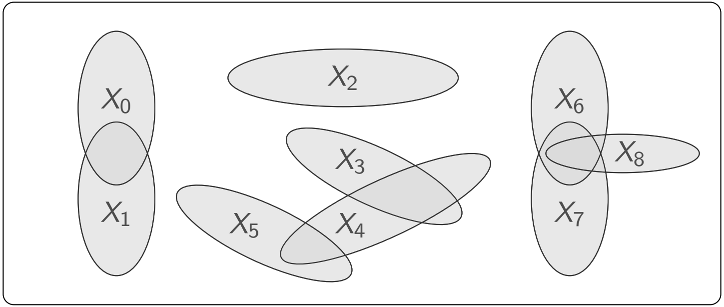
An evidence model $\mathcal{M}$ is **flat** if every scenario on $\mathcal{M}$ has non-empty intersection.

Proposition. The formula $\Box\varphi \rightarrow \langle B \rangle \varphi$ is valid on the class of flat evidence models, but not on the class of all evidence models.

Exercises

1. Prove that $(\Box\varphi \wedge A\psi) \leftrightarrow \Box(\varphi \wedge A\psi)$ is valid on all evidence models.
2. Prove that $B\varphi \rightarrow AB\varphi$ is valid on all uniform evidence models.

Conditional Beliefs on Evidence Models



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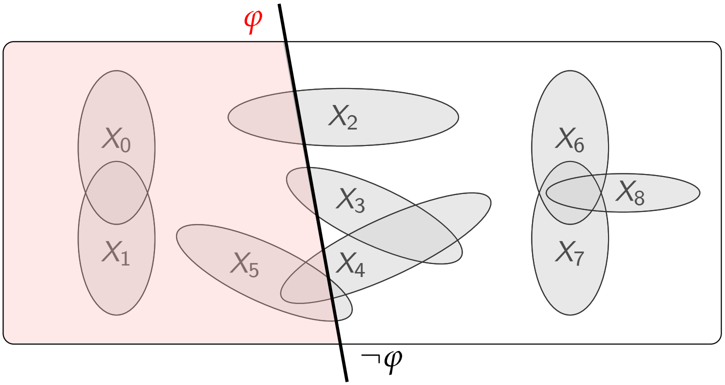

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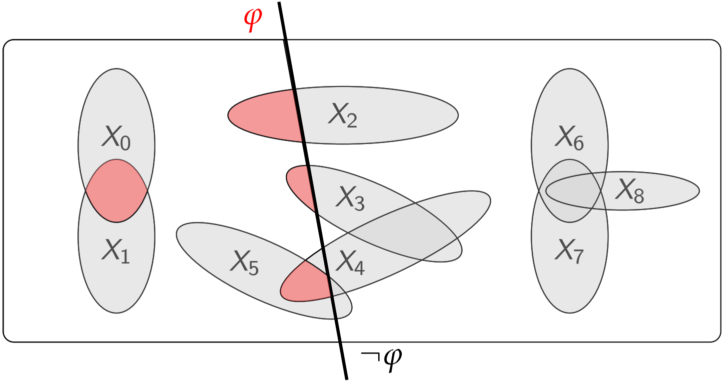

Conditional Beliefs on Evidence Models

$B^{\varphi}\psi$: "the agent believes ψ conditional on φ."

Main idea: Ignore the evidence that is inconsistent with φ.

Relativized w**-scenario**: Suppose that $X \subseteq W$. Given a collection $\mathcal{X} \subseteq \wp(W)$, let $\mathcal{X}^X = \{Y \cap X \mid Y \in \mathcal{X}\}$. We say that a collection $\mathcal{X}$ of subsets of W has the **finite intersection property relative to** X (X-f.i.p.) if, $\mathcal{X}^X$ as the f.i.p. and is maximal if $\mathcal{X}^X$ is.

- $\mathcal{M}, w \models B^{\varphi}\psi$ iff for each maximal φ-f.i.p. $\mathcal{X} \subseteq E(w)$, for each $v \in \bigcap \mathcal{X}^{\varphi}$, $\mathcal{M}, v \models \psi$

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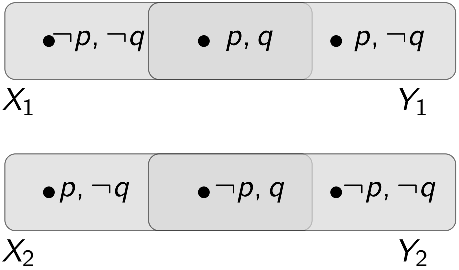

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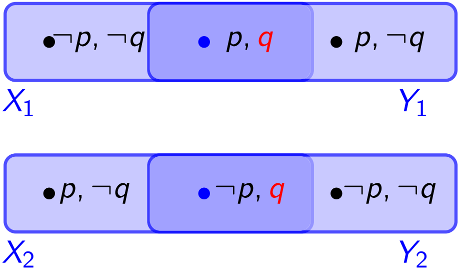

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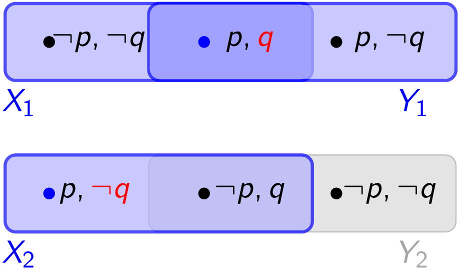

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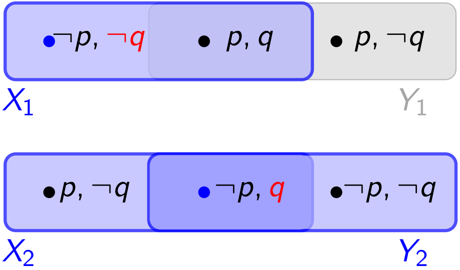

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