Modal Logic:
Incompleteness and Non-Normal Modal Logics

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Basic modal language $\mathcal{L}$: $p \mid \neg\varphi \mid (\varphi \wedge \psi) \mid \Diamond\varphi$
where $p \in \mathsf{At}$ (the set of atomic propositions)

Frame: $\mathcal{F} = \langle W, R \rangle$ where $W \neq \emptyset$ and $R \subseteq W \times W$

Model: $\mathcal{M} = \langle W, R, V \rangle$ where $\langle W, R \rangle$ is a frame and $V : \mathsf{At} \to \wp(W)$

Truth:

- $\mathcal{M}, w \models p$ iff $w \in V(p)$
- $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$
- $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \Diamond\varphi$ iff there is a $v \in W$ such that wRv and $\mathcal{M}, v \models \varphi$

$\mathcal{F} \models \varphi$ when for all $\mathcal{M}$ based on $\mathcal{F}$ and states w, $\mathcal{M}, w \models \varphi$

Given a $\mathcal{M} = \langle W, R, V \rangle$, let $[\![\cdot]\!]_{\mathcal{M}} : \mathcal{L} \to \wp(W)$ be the map where:

$$[\![\varphi]\!]_{\mathcal{M}} = \{w \mid \mathcal{M}, w \models \varphi\}$$

$$[\![p]\!]_{\mathcal{M}} = V(p)$$
$$[\![\neg\varphi]\!]_{\mathcal{M}} = W \setminus [\![\varphi]\!]_{\mathcal{M}}$$
$$[\![\varphi \vee \psi]\!]_{\mathcal{M}} = [\![\varphi]\!]_{\mathcal{M}} \cup [\![\psi]\!]_{\mathcal{M}}$$
$$[\![\varphi \wedge \psi]\!]_{\mathcal{M}} = [\![\varphi]\!]_{\mathcal{M}} \cap [\![\psi]\!]_{\mathcal{M}}$$
$$[\![\Diamond\varphi]\!]_{\mathcal{M}} = R^{-1}([\![\varphi]\!]_{\mathcal{M}}) = \{w \mid \text{there is } x \in [\![\varphi]\!]_{\mathcal{M}} \text{ such that } w \, R \, x\}$$

A logic $\mathbf{L} \subseteq \mathcal{L}$ is a **normal modal logic** if

- $\mathbf{L}$ contains all tautologies of classical propositional logic
- $\mathbf{L}$ is closed under modus ponens
- $\mathbf{L}$ is closed under uniform substitution
- $\mathbf{L}$ is closed under necessitation
- $\Box(p \to q) \to (\Box p \to \Box q) \in \mathbf{L}$

Let $\mathbf{K}$ be the smallest normal modal logic.

Soundness and Completeness: $\mathbf{K}$ is sound and strongly complete with respect to the class of all frames.

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- Are there other semantics for the basic modal language?

Non-normal modal logics

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(M)	$\Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$
(C)	$(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$
(N)	$\Box\top$
(K)	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
$(Dual)$	$\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$
(Nec)	from $\vdash \varphi$ infer $\vdash \Box\varphi$
(Re)	from $\vdash \varphi \leftrightarrow \psi$ infer $\vdash \Box\varphi \leftrightarrow \Box\psi$

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Logical Omniscience/Knowledge Closure

RM	From $\varphi \rightarrow \psi$, infer $\Box\varphi \rightarrow \Box\psi$
K	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
Nec	From φ, infer $\Box\varphi$
RE	From $\varphi \leftrightarrow \psi$, infer $\Box\varphi \leftrightarrow \Box\psi$

Logical Omniscience/Knowledge Closure

RM From $\varphi \to \psi$, infer $\Box\varphi \to \Box\psi$
closure under logical implication

K $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

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RM	From $\varphi \to \psi$, infer $\Box\varphi \to \Box\psi$ *closure under logical implication*
K	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ *closure under known implication*
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knowledge of all logical validities

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RE From $\varphi \leftrightarrow \psi$, infer $\Box\varphi \leftrightarrow \Box\psi$
closure under logical equivalence

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Nec	From φ, infer $\Box\varphi$ *knowledge of all logical validities*
RE	From $\varphi \leftrightarrow \psi$, infer $\Box\varphi \leftrightarrow \Box\psi$ *closure under logical equivalence*

Logics of High Probability

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Claim: $(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$ is not valid.

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H. Kyburg and C.M. Teng. *The Logic of Risky Knowledge*. Proceedings of WoLLIC (2002).

A. Herzig. *Modal Probability, Belief, and Actions*. Fundamenta Informaticae (2003).

Deontic Logic

$\Box\varphi$ mean "*it is obliged that* φ."

$$\frac{\varphi \to \psi}{\Box\varphi \to \Box\psi}$$

J. Forrester. *Paradox of Gentle Murder*. 1984.

L. Goble. *Murder Most Gentle: The Paradox Deepens*. 1991.

Deontic Logic

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1. Jones murders Smith
2. Jones ought not to murder Smith

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Deontic Logic

$\Box\varphi$ mean "*it is obliged that* φ."

1. Jones murders Smith
2. Jones ought not to murder Smith
3. If Jones murders Smith, then Jones ought to murder Smith gently

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✓ Jones murders Smith

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✓ If Jones murders Smith, then Jones ought to murder Smith gently

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$\Rightarrow$ If Jones murders Smith gently, then Jones murders Smith.

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5. If Jones murders Smith gently, then Jones murders Smith.

✓ If Jones ought to murder Smith gently, then Jones ought to murder Smith

7. Jones ought to murder Smith

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4. Jones ought to murder Smith gently
5. If Jones murders Smith gently, then Jones murders Smith.
6. If Jones ought to murder Smith gently, then Jones ought to murder Smith
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Abilities

$Abl_i\varphi$: *i* has the ability to *see to it that* φ is true
(alternatively, *i* has the ability to bring about φ)

What are the core logical principles?

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5. $Abl_i(\varphi \wedge \psi) \rightarrow (Abl_i\varphi \wedge Abl_i\psi)$
6. $Abl_iAbl_j\varphi \rightarrow Abl_i\varphi$, $Abl_iAbl_i\varphi \rightarrow Abl_i\varphi$

Games: $(Abl_i\varphi \wedge Abl_i\psi) \not\to Abl_i(\varphi \wedge \psi)$

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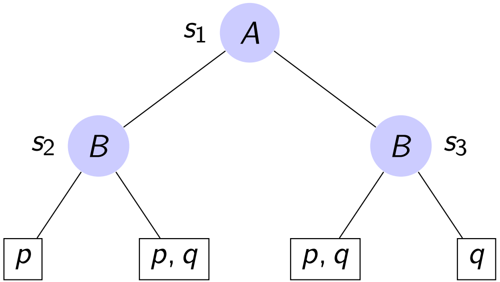

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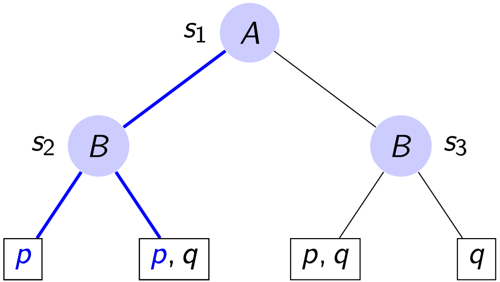

$$s_1 \models Abl_A p$$

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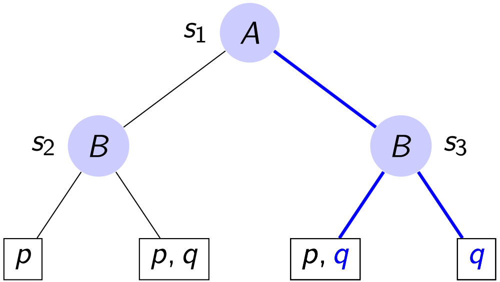

$$s_1 \models Abl_A p \wedge Abl_A q$$

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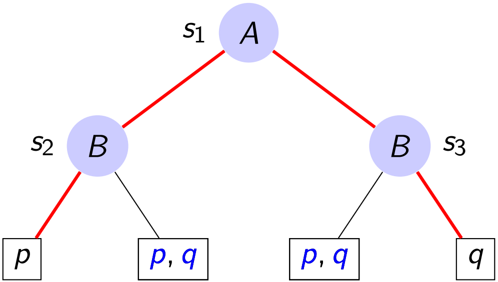

$$s_1 \models Abl_A p \wedge Abl_A q \wedge \neg Abl_A(p \wedge q)$$

Games: $(Abl_i\varphi \wedge Abl_i\psi) \not\to Abl_i(\varphi \wedge \psi)$

R. Parikh (1985). *The Logic of Games and its Applications*. Annals of Discrete Mathematics.

M. Pauly and R. Parikh (2003). *Game Logic — An Overview*. Studia Logica.

J. van Benthem (2014). *Logic in Games*. The MIT Press.

$\varphi \not\to Abl_i\varphi$

Suppose an agent (call her Ann) is throwing a dart and she is not a very good dart player, but she just happens to throw a bull's eye.

$\varphi \not\to Abl_i\varphi$

Suppose an agent (call her Ann) is throwing a dart and she is not a very good dart player, but she just happens to throw a bull's eye.

Intuitively, we do not want to say that Ann has the *ability* to throw a bull's eye even though it happens to be true.

$\mathit{Abl}_i(\varphi \vee \psi) \not\to \mathit{Abl}_i\varphi \vee \mathit{Abl}_i\psi$

Continuing with this example, suppose that Ann has the ability to hit the dart board, but has no other control over the placement of the dart.

$Abl_i(\varphi \lor \psi) \not\to Abl_i\varphi \lor Abl_i\psi$

Continuing with this example, suppose that Ann has the ability to hit the dart board, but has no other control over the placement of the dart.

Now, when she throws the dart, as a matter of fact, it will either hit the top half of the board or the bottom half of the board.

$Abl_i(\varphi \vee \psi) \not\to Abl_i\varphi \vee Abl_i\psi$

Continuing with this example, suppose that Ann has the ability to hit the dart board, but has no other control over the placement of the dart.

Now, when she throws the dart, as a matter of fact, it will either hit the top half of the board or the bottom half of the board.

Since, Ann has the ability to hit the dart board, she has the ability to either hit the top half of the board or the bottom half of the board.

$Abl_i(\varphi \vee \psi) \not\to Abl_i\varphi \vee Abl_i\psi$

Continuing with this example, suppose that Ann has the ability to hit the dart board, but has no other control over the placement of the dart.

Now, when she throws the dart, as a matter of fact, it will either hit the top half of the board or the bottom half of the board.

Since, Ann has the ability to hit the dart board, she has the ability to either hit the top half of the board or the bottom half of the board.

However, intuitively, it seems true that Ann does not have the ability to hit the top half of the dart board, and also she does not have the ability to hit the bottom half of the dart board.

Abilities

$Abl_i\varphi$: agent i has the ability to bring about (see to it that) φ is true

What are core logical principles? Depends very much on the intended "application" and how actions are represented...

1. $Abl_i\varphi \to \varphi$ (or $\varphi \to Abl_i\varphi$)
2. $\neg Abl_i\top$
3. $(Abl_i\varphi \wedge Abl_i\psi) \to Abl_i(\varphi \wedge \psi)$
4. $Abl_i(\varphi \vee \psi) \to (Abl_i\varphi \vee Abl_i\psi)$
5. $Abl_i(\varphi \wedge \psi) \to (Abl_i\varphi \wedge Abl_i\psi)$
6. $Abl_iAbl_j\varphi \to Abl_i\varphi$, $Abl_iAbl_i\varphi \to Abl_i\varphi$

On the Logic of Ability

$$Abl_i \top$$

$$\varphi \rightarrow Abl_i \varphi$$

$$(Abl_i \varphi \wedge Abl_i \psi) \rightarrow Abl_i(\varphi \wedge \psi)$$

$$Abl_i(\varphi \vee \psi) \rightarrow (Abl_i \varphi \vee Abl_i \psi)$$

On the Logic of Ability

$\neg Abl_i \top$

$\varphi \not\to Abl_i \varphi$

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On the Logic of Ability

$\neg Abl_i \top$

$\Box\top$ is valid in the class of all frames, $\Diamond\top$ is valid on the class of serial frames

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$\varphi \rightarrow \Diamond\varphi$ is valid in the class of reflexive frames

$(Abl_i \varphi \wedge Abl_i \psi) \not\rightarrow Abl_i(\varphi \wedge \psi)$

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$(\Box \varphi \wedge \Box \psi) \to \Box (\varphi \wedge \psi)$ is valid in the class of all frames

$Abl_i (\varphi \vee \psi) \not\to (Abl_i \varphi \vee Abl_i \psi)$

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$Abl_i(\varphi \vee \psi) \not\to (Abl_i \varphi \vee Abl_i \psi)$

$\Diamond(\varphi \vee \psi) \to (\Diamond \varphi \vee \Diamond \psi)$ is valid in the class of all frames

M. Brown. *On the Logic of Ability*. Journal of Philosophical Logic, Vol. 17, pp. 1 - 26, 1988.

D. Elgesem. *The modal logic of agency*. Nordic Journal of Philosophical Logic 2(2), 1 - 46, 1997.

G. Governatori and A. Rotolo. *On the Axiomatisation of Elgesem's Logic of Agency and Ability*. Journal of Philosophical Logic, 34, pgs. 403 - 431 (2005).

A Minimal Logic of Abilities

$C\varphi$ means "the agent is capable of realizing φ"

$E\varphi$ means "the agent does bring about φ"

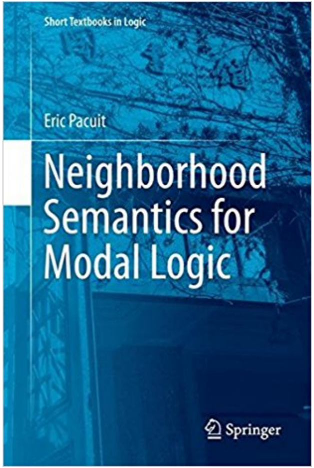
A Minimal Logic of Abilities

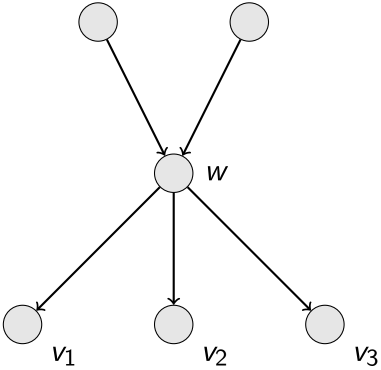
$C\varphi$ means "the agent is capable of realizing φ"

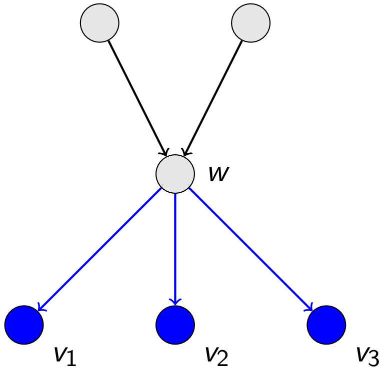
$E\varphi$ means "the agent does bring about φ"

1. All propositional tautologies
2. $\neg C\top$
3. $E\varphi \wedge E\psi \to E(\varphi \wedge \psi)$
4. $E\varphi \to \varphi$
5. $E\varphi \to C\varphi$
6. Modus Ponens plus from $\varphi \leftrightarrow \psi$ infer $E\varphi \leftrightarrow E\psi$ and from $\varphi \leftrightarrow \psi$ infer $C\varphi \leftrightarrow C\psi$

Neighborhood Semantics for Modal Logic




w
v1
v2
v3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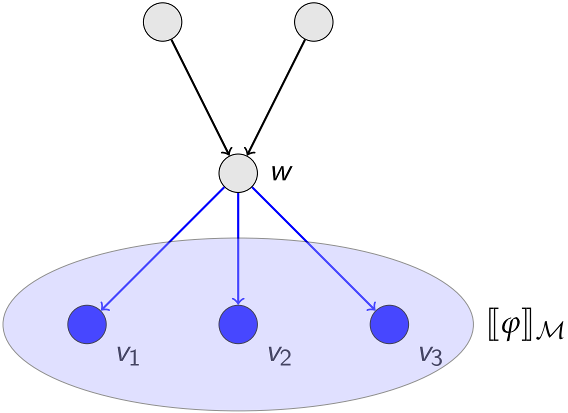
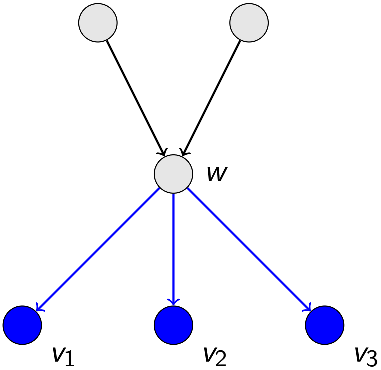


$$\mathcal{M}, w \models \Box\varphi \text{ iff } R(w) \subseteq [\![\varphi]\!]_{\mathcal{M}}$$

...**the** neighborhood of w is
contained in the truth-set of φ



$$\mathcal{M}, w \models \boxplus\varphi \text{ iff } R(w) = [\![\varphi]\!]_{\mathcal{M}}$$

...**the** neighborhood of w **is** the truth-set of φ

$w \models \Box\varphi$ if the truth set of φ is a neighborhood of w

$$w \models \Box\varphi \qquad \text{if the truth set of } \varphi \text{ is a neighborhood of } w$$

What does it mean to be a neighborhood?

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neighborhood in some topology.

J. McKinsey and A. Tarski. *The Algebra of Topology*. 1944.

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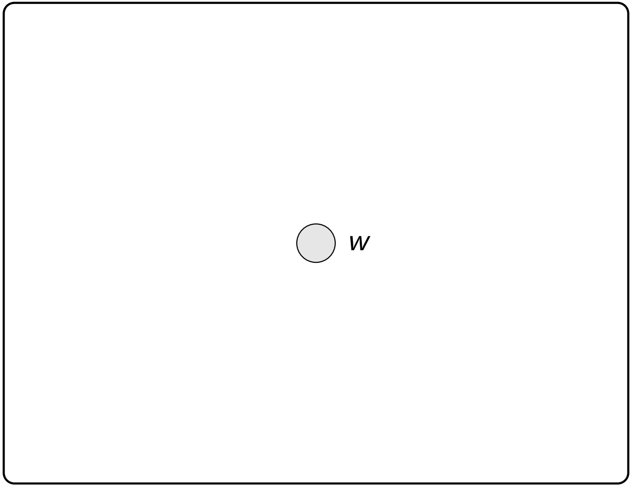
an element of some distinguished collection of sets

D. Scott. *Advice on Modal Logic*. 1970.

R. Montague. *Pragmatics*. 1968.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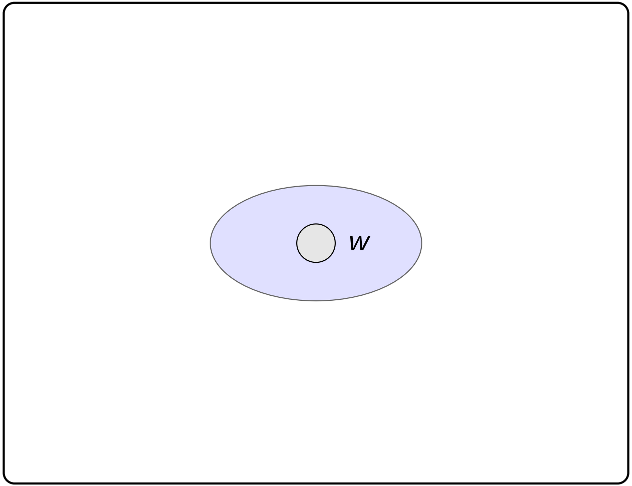
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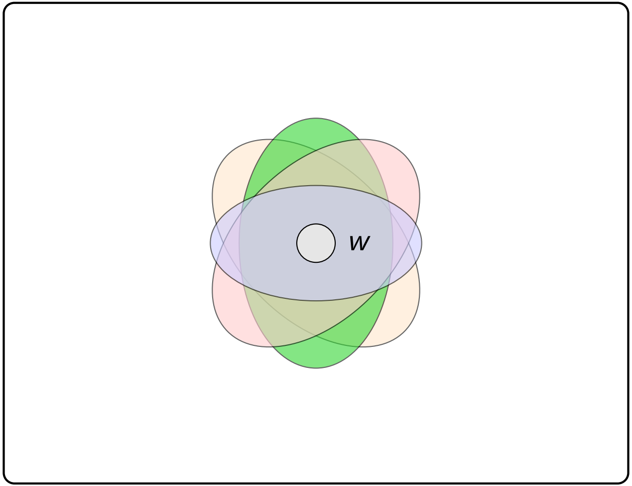
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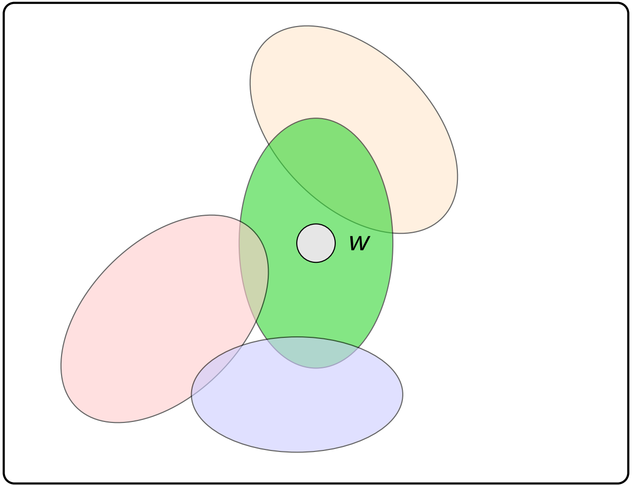
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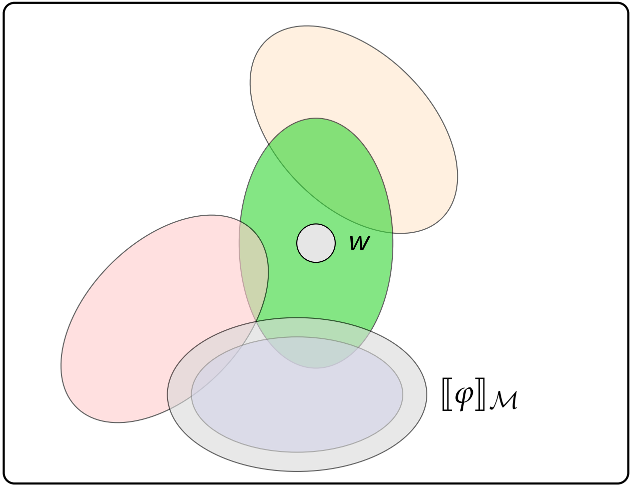


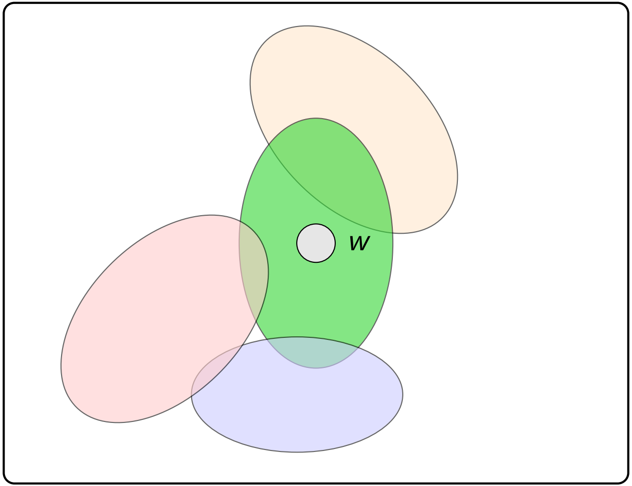
$\mathcal{M}, w \models \Box\varphi$ iff **there is** a neighborhood of w **contained in** $[\![\varphi]\!]_{\mathcal{M}}$

Relational model: $\langle W, R, V\rangle$ where $R : W \to \wp(W)$

$w \models \Box\varphi$ iff $R(w) \subseteq [\![\varphi]\!]$

Neighborhood model: $\langle W, N, V\rangle$ where $N : W \to \wp(\wp(W))$

$w \models \Box\varphi$ iff there is a $X \in N(w)$ such that $X \subseteq [\![\varphi]\!]$



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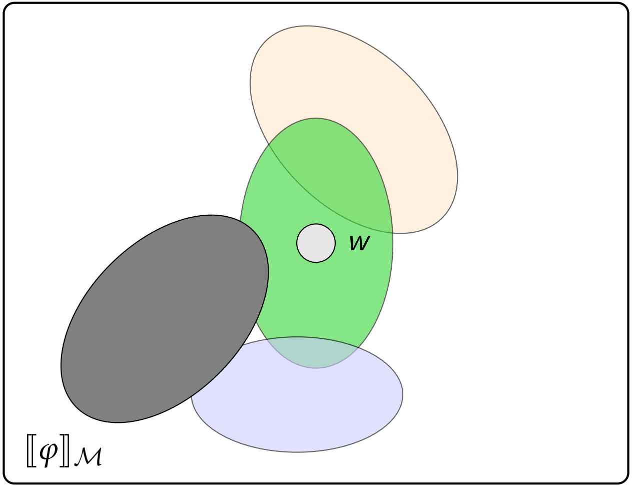

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Neighborhood Frames

Let W be a non-empty set of states.

Any function $N : W \to \wp(\wp(W))$ is called a neighborhood function

A pair $\langle W, N \rangle$ is a called a neighborhood frame if W a non-empty set and N is a neighborhood function.

A neighborhood model based on $\mathfrak{F} = \langle W, N \rangle$ is a tuple $\langle W, N, V \rangle$ where $V : \text{At} \to \wp(W)$ is a valuation function.

Truth in a Model

- $\mathfrak{M}, w \models p$ iff $w \in V(p)$
- $\mathfrak{M}, w \models \neg\varphi$ iff $\mathfrak{M}, w \not\models \varphi$
- $\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$

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- $\mathfrak{M}, w \models \Box\varphi$ iff $[\![\varphi]\!]_{\mathfrak{M}} \in N(w)$
- $\mathfrak{M}, w \models \Diamond\varphi$ iff $W - [\![\varphi]\!]_{\mathfrak{M}} \notin N(w)$

where $[\![\varphi]\!]_{\mathfrak{M}} = \{w \mid \mathfrak{M}, w \models \varphi\}$.

Let $N : W \to \wp\wp W$ be a neighborhood function and define $m_N : \wp W \to \wp W$:

$$\text{for } X \subseteq W,\ m_N(X) = \{w \mid X \in N(w)\}$$

1. $[\![p]\!]_{\mathfrak{M}} = V(p)$ for $p \in \mathsf{At}$
2. $[\![\neg\varphi]\!]_{\mathfrak{M}} = W - [\![\varphi]\!]_{\mathfrak{M}}$
3. $[\![\varphi \wedge \psi]\!]_{\mathfrak{M}} = [\![\varphi]\!]_{\mathfrak{M}} \cap [\![\psi]\!]_{\mathfrak{M}}$
4. $[\![\Box\varphi]\!]_{\mathfrak{M}} = m_N([\![\varphi]\!]_{\mathfrak{M}})$
5. $[\![\Diamond\varphi]\!]_{\mathfrak{M}} = W - m_N(W - [\![\varphi]\!]_{\mathfrak{M}})$

Detailed Example

Suppose $W = \{w, s, v\}$ is the set of states and define a neighborhood model $\mathfrak{M} = \langle W, N, V\rangle$ as follows:

- $N(w) = \{\{s\}, \{v\}, \{w, v\}\}$
- $N(s) = \{\{w, v\}, \{w\}, \{w, s\}\}$
- $N(v) = \{\{s, v\}, \{w\}, \emptyset\}$

Further suppose that $V(p) = \{w, s\}$ and $V(q) = \{s, v\}$.

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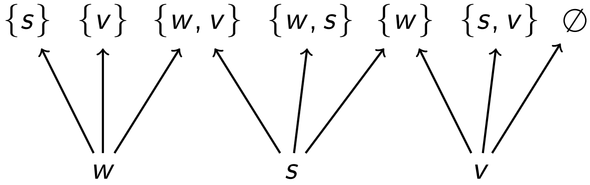

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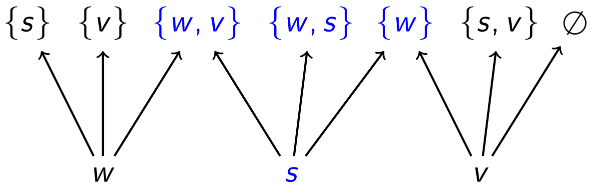

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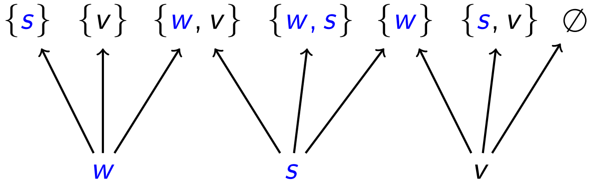

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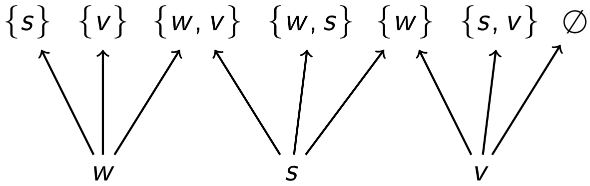

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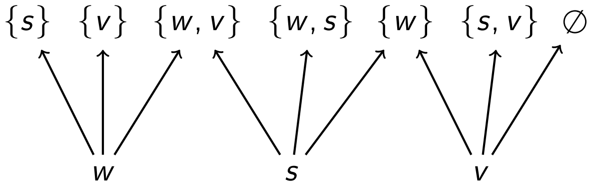

$$\mathfrak{M}, s \models \Box p$$

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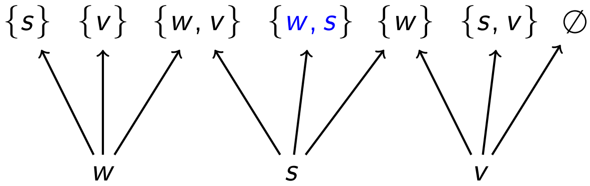

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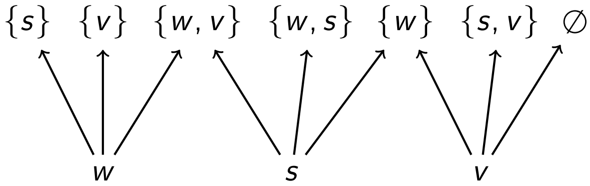

$$\mathfrak{M}, s \models \Diamond p$$

Detailed Example

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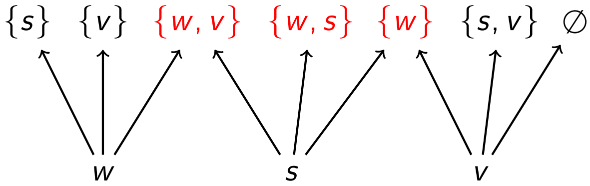

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$$[\![\neg p]\!]_{\mathfrak{M}} = \{v\}$$

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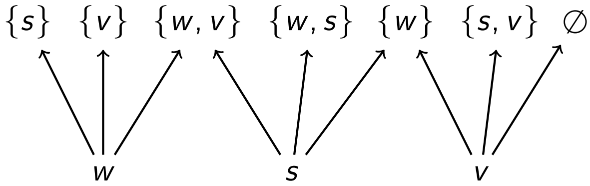

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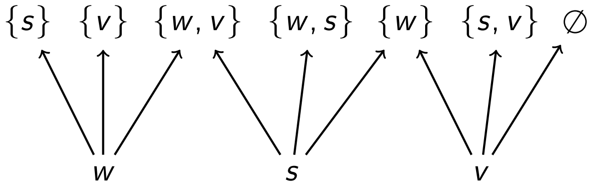

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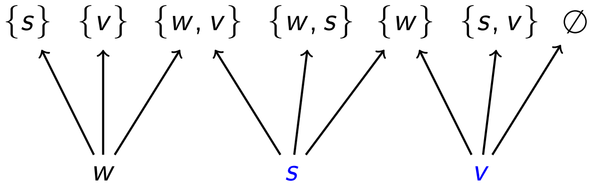

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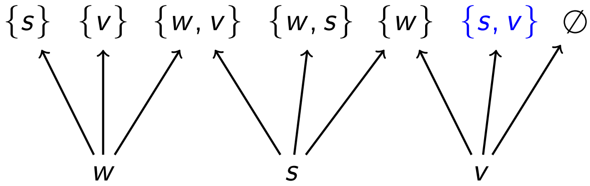

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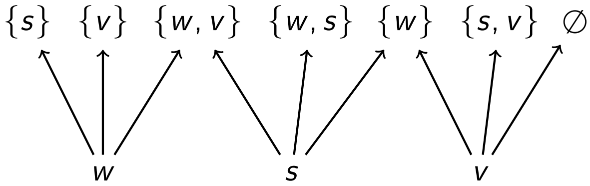

$\mathfrak{M}, w \not\models \Diamond\Box p$ $\qquad$ $\mathfrak{M}, v \models \Box\Diamond p$

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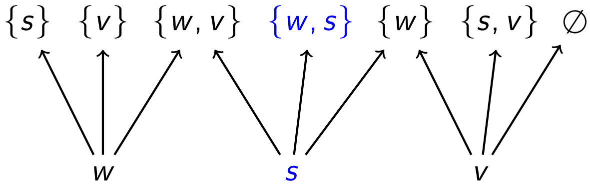

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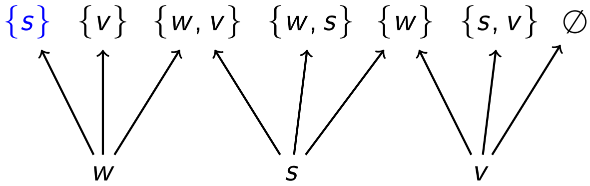

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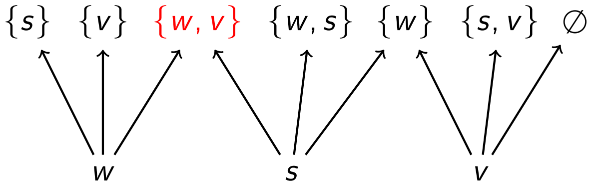

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