Modal Logic: Logics of Knowledge and Belief

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Doxastic Logic: Models

Model: $\langle W, R, V \rangle$

States/possible worlds: $W \neq \emptyset$

Quasi-partitions: $R \subseteq W \times W$ is serial, transitive and Euclidean

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- *serial*: for all $w \in W$, there is a $v \in W$ such that $w \, R \, v$
- *transitive*: for all $w, v, u \in W$, if $w \, R \, v$ and $v \, R \, u$, then $w \, R \, u$
- *Euclidean*: for all $w, v, u \in W$, if $w \, R \, v$ and $w \, R \, u$, then $v \, R \, u$

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Valuation function: $V : \mathrm{At} \to \wp(W)$, where At is a set of atomic propositions.

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Boolean connectives:

- $\mathcal{M}, w \models p$ iff $w \in V(p)$
- $\mathcal{M}, w \models \neg\varphi$ iff it is not the case that $\mathcal{M}, w \models \varphi$
- $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$

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$\{v \mid w\,R\,v\}$ $\qquad$ $\{v \mid \mathcal{M}, w \models \varphi\}$

Doxastic Logic: **KD45**

K	$B(\varphi \rightarrow \psi) \rightarrow (B\varphi \rightarrow B\psi)$
D	$B\varphi \rightarrow \neg B \neg \varphi$
4	$B\varphi \rightarrow BB\varphi$
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The logic **KD45** adds the above axiom schemes to an axiomatization of classical propositional logic with the rules Modus Ponens, Substitution of Equivalents, and Necessitation (from φ infer $B\varphi$).

KD45 is sound and strongly complete with respect to all quasi-partition frames.

Exercise: Show that the following axiom schemes and rules are valid on quasi-partition models and are theorems of **KD45**:

- agglomeration: $(B\varphi \wedge B\psi) \to B(\varphi \wedge \psi)$
- consistency: $\neg B\bot$
- monotonicity: From $\varphi \to \psi$ infer $B\varphi \to B\psi$

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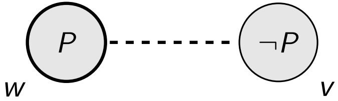
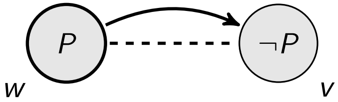
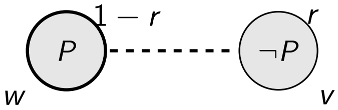
  $B(B\varphi \to \varphi)$
- correctness of own beliefs:
  $B\neg B\varphi \to \neg B\varphi$
  for all w, there is a v such that $w\,R\,v$ and for all z if $v\,R\,z$ then $w\,R\,z$
  $BB\varphi \to B\varphi$
  density: for all w and v if $w\,R\,v$ then there is a z such that $w\,R\,z$ and $z\,R\,v$



Ann does not know that P



Ann does not know that P, but she believes that $\neg P$



Ann does not know that P, but she believes that $\neg P$ is true to degree r.

Combining Logics of Knowledge and Belief

$\mathcal{M} = \langle W, \sim, R, V \rangle$ where

- $W \neq \emptyset$ is a set of states;
- each $\sim$ is an equivalence relation on W;
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- V is a valuation function.

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- K is **S5**

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- $B\varphi \to BK\varphi$? "positive certainty"

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- $K\varphi \rightarrow B\varphi$? "knowledge implies belief"
- $B\varphi \rightarrow BK\varphi$? "positive certainty"
- $B\varphi \rightarrow KB\varphi$? "strong introspection"

An Issue - Negative Introspection and Positive Certainty

- Suppose that p is something you are certain of (you *believe* it with probability one), but is false: $\neg p \wedge Bp$

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- $Bp \rightarrow BKp$
- $\neg p \rightarrow \neg Kp \rightarrow K\neg Kp \rightarrow B\neg Kp$
- So, $BKp \wedge B\neg Kp$ also holds, but this contradictions $B\varphi \rightarrow \neg B\neg\varphi$.

Defining Beliefs from Knowledge

R. Stalnaker (2006). *On logics of knowledge and belief*. Philosophy Studies,128,169-199.

A. Baltag, N. Bezhanishvili, A. Özgün, and S. Smets (2019). *A Topological Approach to Full Belief*. Journal of Philosophical Logic, 48(2), pp. 205 - 244.

A. Bjorndahl and A. Özgün (2020). *Logic and Topology for Knowledge, Knowability, and Belief*. The Review of Symbolic Logic, 13(4), pp. 748-775.

Stalnaker's Axioms

Stalnaker bases his analysis on a conception of belief as 'subjective certainty': From the point of the agent in question, her belief is subjectively indistinguishable from her knowledge.

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Bi-modal language of knowledge and belief: $p \mid \neg\varphi \mid \varphi \wedge \psi \mid K\varphi \mid B\psi$
Define $\langle K \rangle\varphi$ as $\neg K \neg\varphi$ and $\langle B \rangle\varphi$ as $\neg B \neg\varphi$

Stalnaker's Axioms

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KB	$K\varphi \to B\varphi$
FB	$B\varphi \to BK\varphi$

Proposition (Stalnaker). The following equivalence is a theorem of the propositional modal logic that contains the previous axiom schemas (with Modus Ponens and Necessitation for both K and B):

$$B\varphi \leftrightarrow \langle K \rangle K\varphi$$

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Moreover, all of the axioms of **KD45** and the (.2)-axiom $\langle K \rangle K\varphi \rightarrow K\langle K \rangle\varphi$ are provable.

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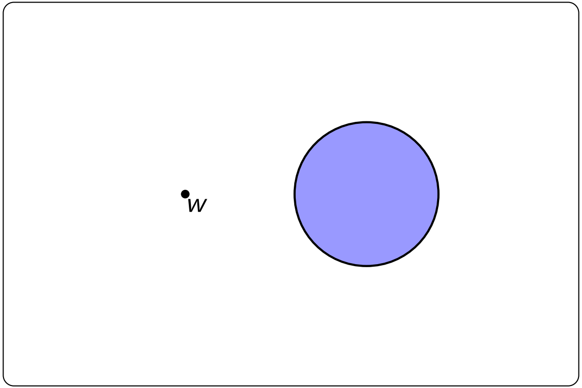
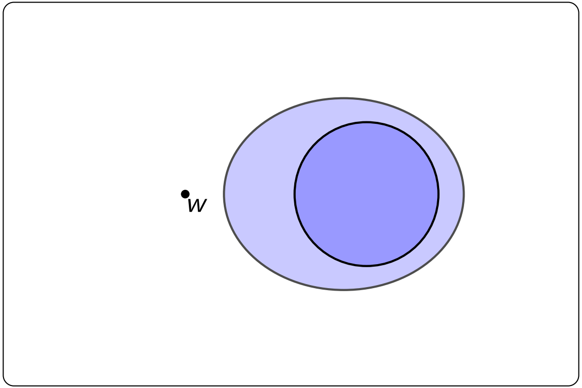
This means that we can take the logic of knowledge to be **S4.2** (the axioms K, T, 4 and .2) and *define* full belief as above (i.e., as the ‘epistemic possibility of knowledge’).



- The agent's beliefs (soft information—-the states consistent with what the agent believes)



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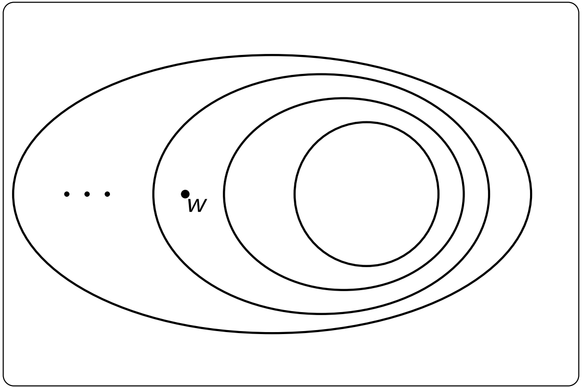

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Sphere Models

Let W be a set of states, A set $\mathcal{F} \subseteq \wp(W)$ is called a system of spheres provided:

- For each $S, S' \in \mathcal{F}$, either $S \subseteq S'$ or $S' \subseteq S$
- For any $P \subseteq W$ there is a smallest $S \in \mathcal{F}$ (according to the subset relation) such that $P \cap S \neq \emptyset$
- The spheres are non-empty $\bigcap \mathcal{F} \neq \emptyset$ and cover the entire information cell $\bigcup \mathcal{F} = W$

Let $\mathcal{F}$ be a system of spheres on W: for $w, v \in W$, let

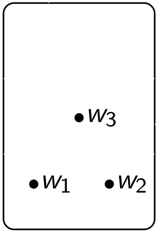
$$w \preceq_{\mathcal{F}} v \text{ iff for all } S \in \mathcal{F}, \text{ if } v \in S \text{ then } w \in S$$

Then, $\preceq_{\mathcal{F}}$ is reflexive, transitive, and well-founded.

$w \preceq_{\mathcal{F}} v$ means that no matter what the agent learns in the future, as long as world v is still consistent with her beliefs and w is still epistemically possible, then w is also consistent with her beliefs.

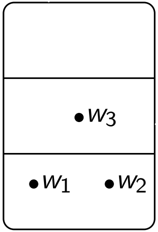
Belief Revision via Plausibility

- $W = \{w_1, w_2, w_3\}$



Belief Revision via Plausibility

- $W = \{w_1, w_2, w_3\}$
- $w_1 \preceq w_2$ and $w_2 \preceq w_1$ (w_1 and w_2 are equi-plausbile)
- $w_1 \prec w_3$ ($w_1 \preceq w_3$ and $w_3 \not\preceq w_1$)
- $w_2 \prec w_3$ ($w_2 \preceq w_3$ and $w_3 \not\preceq w_2$)



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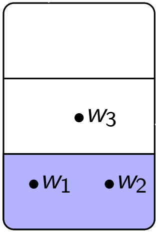
- $w_2 \prec w_3$ ($w_2 \preceq w_3$ and $w_3 \not\preceq w_2$)
- $\{w_1, w_2\} \subseteq Min_{\preceq}([w_i])$



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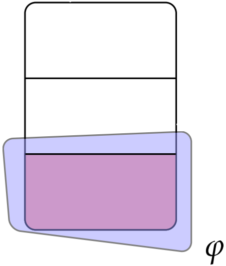
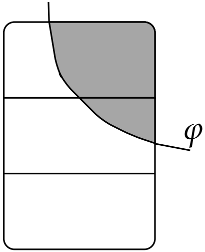


Belief: $B\varphi$

$$Min_{\preceq}(W) \subseteq [\![\varphi]\!]_{\mathcal{M}}$$

Belief Revision via Plausibility



Conditional Belief: $B^{\varphi}\psi$

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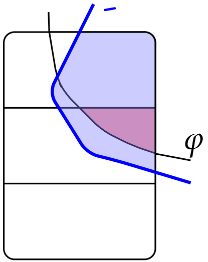

Conditional Belief: $B^{\varphi}\psi$

$$Min_{\preceq}([\![\varphi]\!]_{\mathcal{M}}) \subseteq [\![\psi]\!]_{\mathcal{M}}$$