# Modal Logic: Logics of Knowledge and Belief

Eric Pacuit, University of Maryland

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# The "Problem" of Logical Omniscience

The rule

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Given this, there are two ways to view $K_i$: as representing either the idealized (implicit, "virtual") knowledge of ordinary agents, or the ordinary knowledge of idealized agents. For discussion, see

R. Stalnaker (1991). *The Problem of Logical Omniscience, I*. Synthese.

R. Stalnaker (2006). *On Logics of Knowledge and Belief*. Philosophical Studies.

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reflects so-called (*synchronic*) *logical omniscience*: the agent knows (at time $t$) all the consequences of what she knows (at $t$).

There is now a large literature on alternative frameworks for representing the knowledge of agents with bounded rationality, who do not always "put two and two together" and therefore lack the logical omniscience reflected by $\mathsf{RK}_i$. See, for example:

J. Y. Halpern and R. Pucella. 2011. *Dealing with Logical Omniscience: Expressiveness and Pragmatics*. Artificial Intelligence.

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*Non-Normal Modal Logics*

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- *Algorithmic knowledge*: $\mathcal{M}, w \models K_i\varphi$ iff $\mathsf{A}_i(w, \varphi) = \mathrm{Yes}$
- *Impossible worlds*: $\mathcal{M}, w \models K_i\varphi$ iff if $w \in N$, then for all $v \in W$, if $wR_iv$ and $v \in N$ then $\mathcal{M}, v \models \varphi$; and if $w \notin N$, then $\varphi \in \mathcal{C}_i(w)$

# Justification Logic (1)

$t : \varphi$: "$t$ is a *justification*/*proof* for $\varphi$"

S. Artemov and M. Fitting (2019). *Justification Logic: Reasoning with Reasons*. Cambridge University Press.

S. Artemov and M. Fitting (2020). *Justification logic*. The Stanford Encyclopedia of Philosophy.

S. Artemov. *Explicit provability and constructive semantics (2001)*. The Bulletin of Symbolic Logic 7, pp. 1 - 36.

M. Fitting (2005). *The logic of proofs, semantically*. Annals of Pure and Applied Logic 132, pp. 1 - 25.

# Justification Logic (2)

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- $t : \varphi \rightarrow \varphi$
- $t : (\varphi \rightarrow \psi) \rightarrow (s : \varphi \rightarrow t \cdot s : \psi)$
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- $t : \varphi \to !t : t : \varphi$

**Internalization**: if $\vdash_{JL} \varphi$ then there is a proof polynomial $t$ such that $\vdash_{JL} t : \varphi$
**Realization Theorem**: if $\vdash_{\mathbf{S4}} \varphi$ then there is a proof polynomial $t$ such that $\vdash_{JL} t : \varphi$

# Justification Logic (3)

**Fitting Semantics**: $\mathcal{M} = \langle W, R, \mathcal{E}, V \rangle$

- $W \neq \emptyset$
- $R \subseteq W \times W$
- $\mathcal{E} : W \times \text{ProofTerms} \rightarrow \wp(\mathcal{L}_{JL})$
- $V : \text{At} \rightarrow \wp(W)$

$\mathcal{M}, w \models t : \varphi$ iff for all $v$, if $wRv$ then $\mathcal{M}, v \models \varphi$ and $\varphi \in \mathcal{E}(w, t)$

# Justification Logic (3)

**Monotonicity** For all $w, v \in W$, if $wRv$ then for all proof polynomials $t$, $\mathcal{E}(w, t) \subseteq \mathcal{E}(v, t)$.

**Application** For all proof polynomials $s$, $t$ and for each $w \in W$, if $\varphi \rightarrow \psi \in \mathcal{E}(w, t)$ and $\varphi \in \mathcal{E}(w, s)$, then $\psi \in \mathcal{E}(w, t \cdot s)$

**Proof Checker** For all proof polynomials $t$ and for each $w \in W$, if $\varphi \in \mathcal{E}(w, t)$, then $t : \varphi \in \mathcal{E}(w, !t)$.

**Sum** For all proof polynomials $s, t$ and for each $w \in W$, $\mathcal{E}(w, s) \cup \mathcal{E}(w, t) \subseteq \mathcal{E}(w, s + t)$.

# Doxastic Logic: Models

Model: $\langle W, R, V \rangle$

States/possible worlds: $W \neq \emptyset$

Quasi-partitions: $R \subseteq W \times W$ is serial, transitive and Euclidean

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- *serial*: for all $w \in W$, there is a $v \in W$ such that $w\,R\,v$
- *transitive*: for all $w, v, u \in W$, if $w\,R\,v$ and $v\,R\,u$, then $w\,R\,u$
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Valuation function: $V : \mathrm{At} \to \wp(W)$, where At is a set of atomic propositions.

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Boolean connectives:

- $\mathcal{M}, w \models p$ iff $w \in V(p)$
- $\mathcal{M}, w \models \neg\varphi$ iff it is not the case that $\mathcal{M}, w \models \varphi$
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# Doxastic Logic: **KD45**

| | |
|---|---|
| $K$ | $B(\varphi \to \psi) \to (B\varphi \to B\psi)$ |
| $D$ | $B\varphi \to \neg B \neg \varphi$ |
| 4 | $B\varphi \to BB\varphi$ |
| 5 | $\neg B\varphi \to B\neg B\varphi$ |

# Doxastic Logic: **KD45**

$$\begin{array}{ll} K & B(\varphi \to \psi) \to (B\varphi \to B\psi) \\ D & B\varphi \to \neg B \neg \varphi \\ 4 & B\varphi \to BB\varphi \\ 5 & \neg B\varphi \to B\neg B\varphi \end{array}$$

The logic **KD45** adds the above axiom schemes to an axiomatization of classical propositional logic with the rules Modus Ponens, Substitution of Equivalents, and Necessitation (from $\varphi$ infer $B\varphi$).

**KD45** is sound and strongly complete with respect to all quasi-partition frames.

**Exercise**: Show that the following axiom schemes and rules are valid on quasi-partition models and are theorems of **KD45**:

- agglomeration: $(B\varphi \wedge B\psi) \rightarrow B(\varphi \wedge \psi)$
- consistency: $\neg B\bot$
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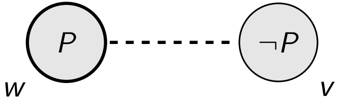
  $B(B\varphi \to \varphi)$
- correctness of own beliefs:
  $B\neg B\varphi \to \neg B\varphi$
  for all $w$, there is a $v$ such that $w\, R\, v$ and for all $z$ if $v\, R\, z$ then $w\, R\, z$
  $BB\varphi \to B\varphi$
  density: for all $w$ and $v$ if $w\, R\, v$ then there is a $z$ such that $w\, R\, z$ and $z\, R\, v$

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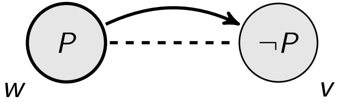
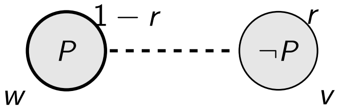



Ann does not know that $P$, but she believes that $\neg P$




Ann does not know that $P$, but she believes that $\neg P$ is true to degree $r$.

# Combining Logics of Knowledge and Belief

$\mathcal{M} = \langle W, \sim, R, V \rangle$ where

- $W \neq \emptyset$ is a set of states;
- each $\sim$ is an equivalence relation on $W$;
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- $B$ is **KD45**
- $K\varphi \rightarrow B\varphi$? "knowledge implies belief"
- $B\varphi \rightarrow BK\varphi$? "positive certainty"
- $B\varphi \rightarrow KB\varphi$? "strong introspection"

# An Issue - Negative Introspection and Positive Certainty

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- $Bp \rightarrow BKp$
- $\neg p \rightarrow \neg Kp \rightarrow K\neg Kp \rightarrow B\neg Kp$
- So, $BKp \wedge B\neg Kp$ also holds, but this contradictions $B\varphi \rightarrow \neg B\neg\varphi$.