Modal Logic: Logics of Knowledge and Belief

Eric Pacuit, University of Maryland

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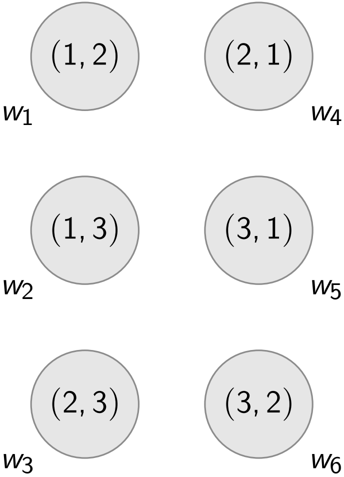

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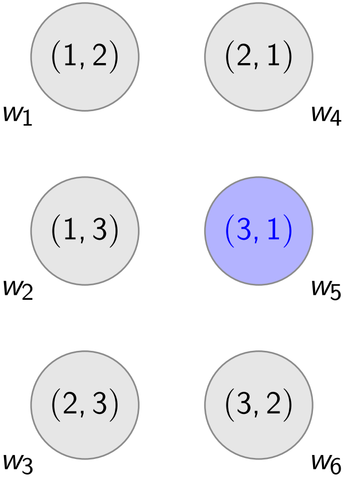

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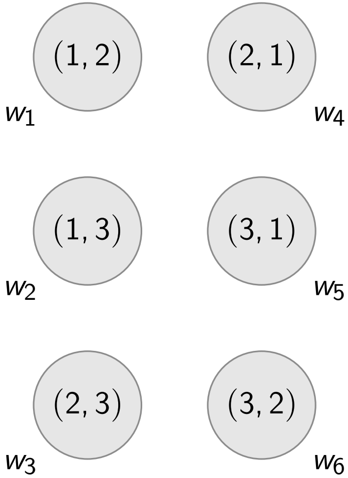

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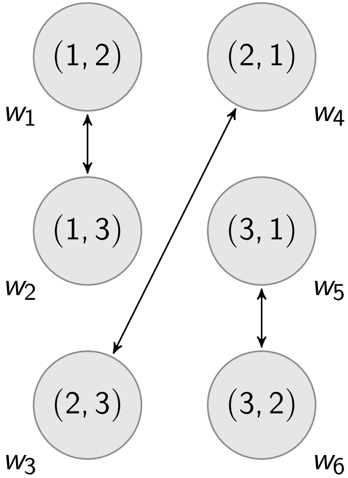

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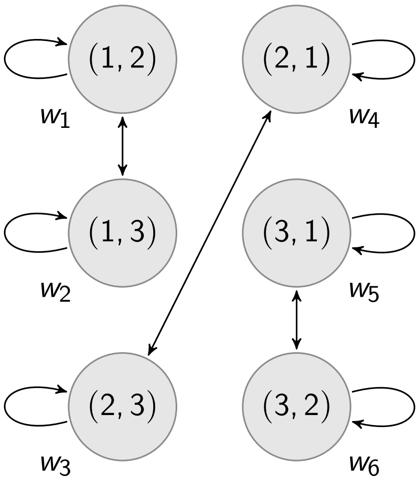

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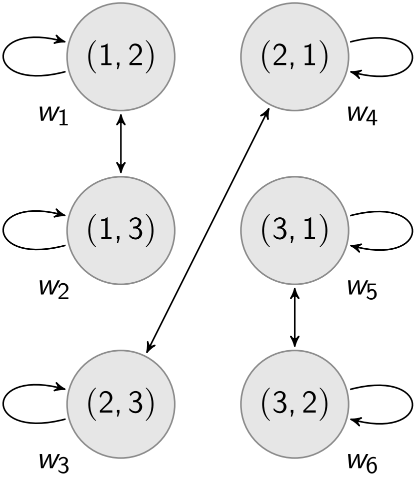

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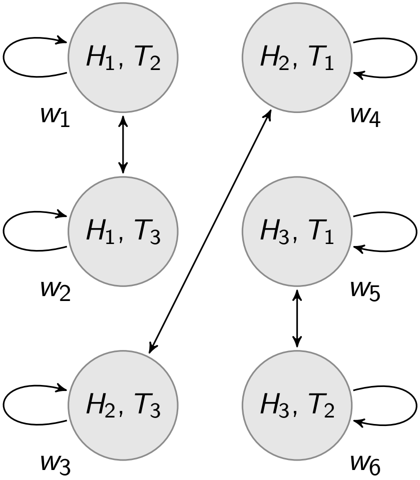

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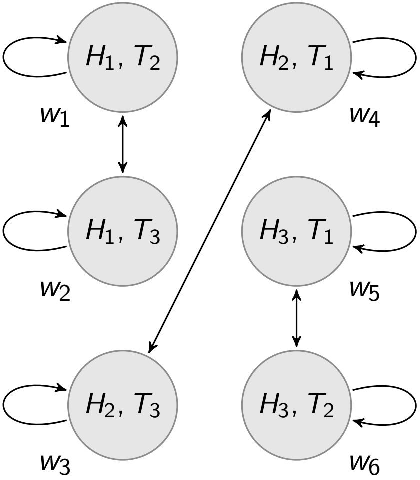

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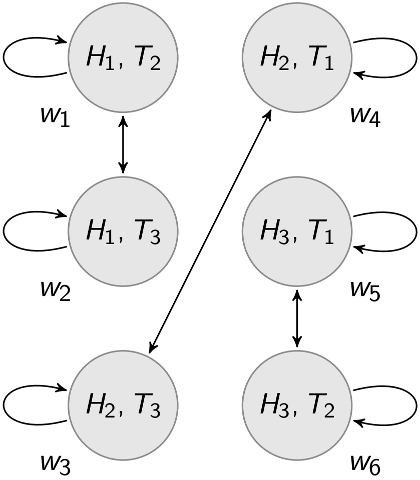

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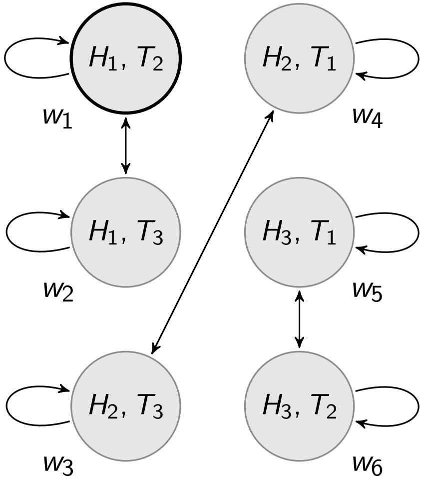

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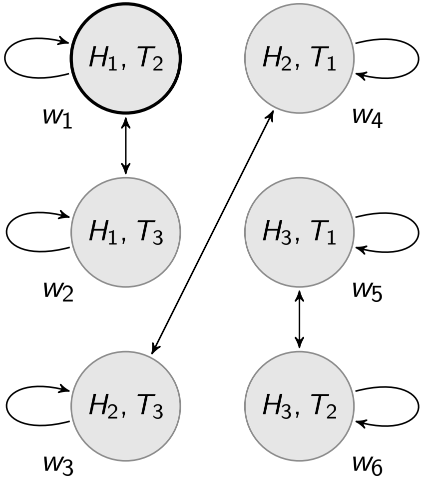

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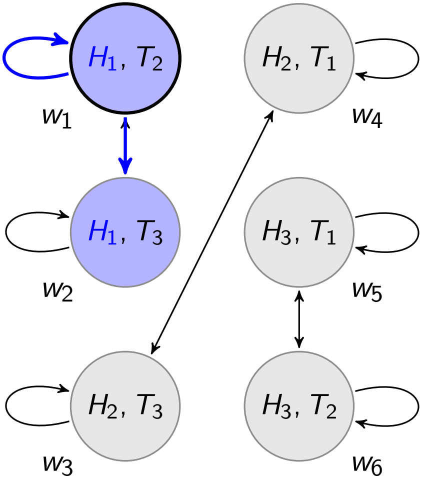

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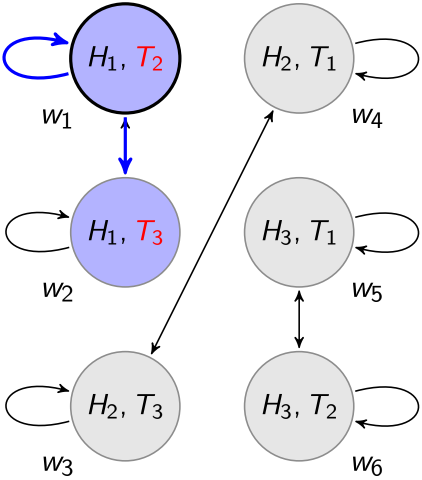

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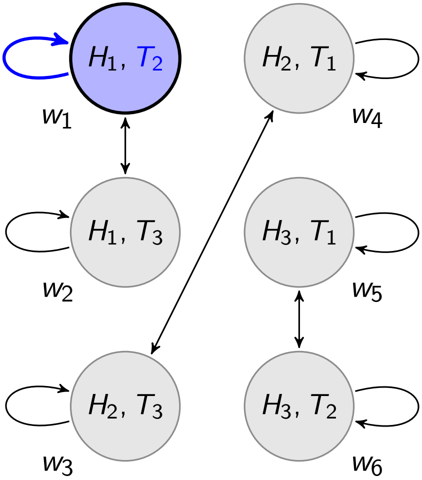

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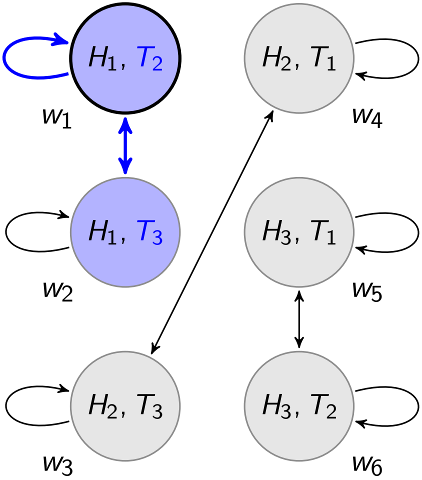

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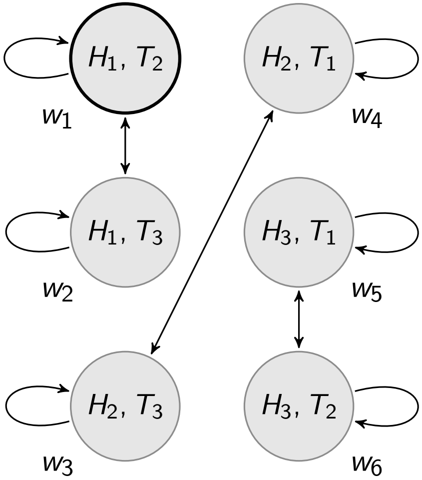

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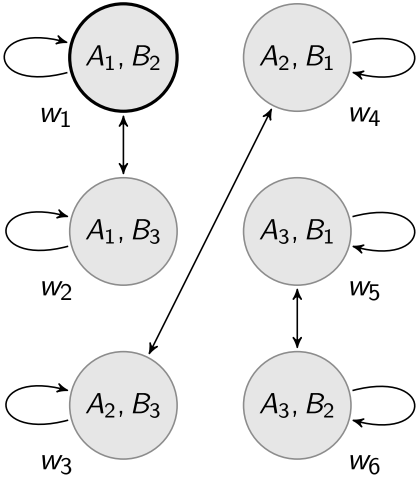

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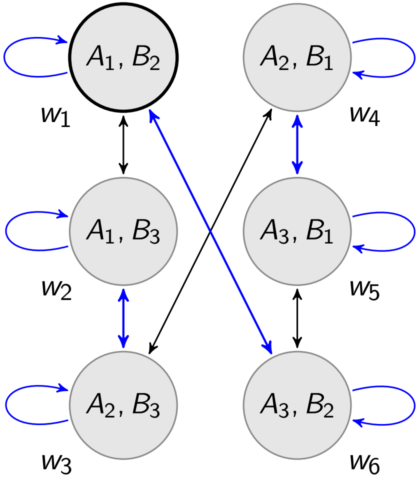

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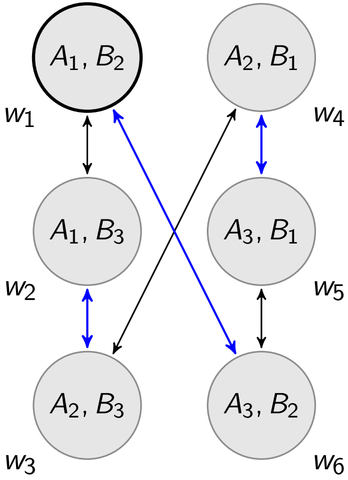

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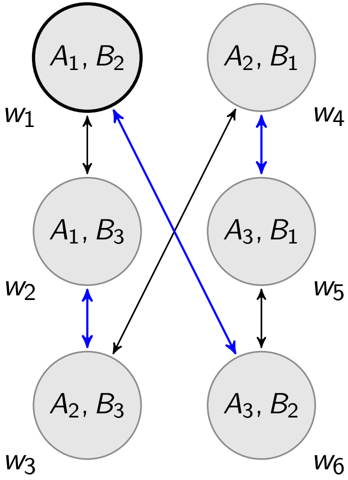

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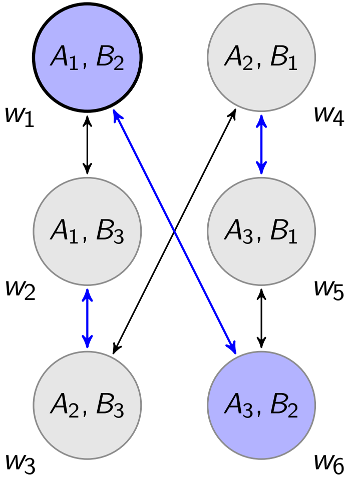

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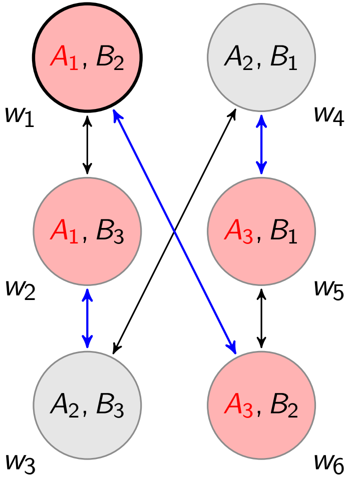

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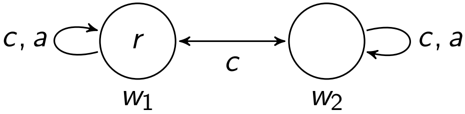

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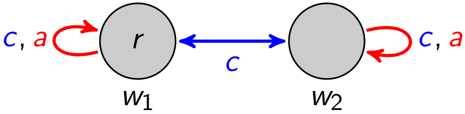

Now suppose that agent c doesn't know whether agent a has left Amsterdam for a vacation. (Let v stand for 'a has left Amsterdam on vacation'.) Agent c knows that if a is not on vacation, then a knows whether it's raining in Amsterdam; but if a is on vacation, then a won't bother to follow the weather.

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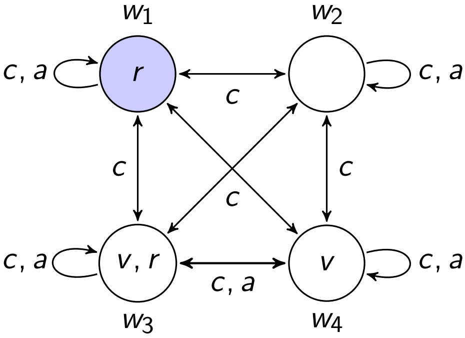

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  - "It is raining"
  - "The talk is at 2PM"
  - "The card on the table is a 7 of Hearts"

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- $K_a\varphi$ is intended to mean "Agent a knows that φ is true".
- The usual definitions for $\rightarrow, \vee, \leftrightarrow$ apply
- Define $L_a\varphi$ (or $\hat{K}_a$) as $\neg K_a \neg\varphi$

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Given $\varphi \in \mathcal{L}$, a Kripke model $\mathcal{M} = \langle W, (R_a)_{a \in Agt}, V \rangle$ and $w \in W$

$\mathcal{M}, w \models \varphi$ means "in $\mathcal{M}$, if the actual state is w, then φ is true"

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- $\mathcal{M}, w \models \varphi \wedge \psi$ if $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
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- ✓ $\mathcal{M}, w \models \varphi \wedge \psi$ if $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
- ✓ $\mathcal{M}, w \models K_a\varphi$ if for each $v \in W$, if wR_av, then $\mathcal{M}, v \models \varphi$
- ✓ $\mathcal{M}, w \models L_a\varphi$ if there exists a $v \in W$ such that wR_av and $\mathcal{M}, v \models \varphi$

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- wR_av if "agent a is in the same *local state* in w and v"

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I.e., $R_a(w) = \{v \mid wR_a v\} \cap [\![\varphi]\!]_{\mathcal{M}} = \{v \mid \mathcal{M}, v \models \varphi\} \neq \emptyset$

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- ~~$L_a\varphi$: "Agent a thinks that φ might be true."~~
- ~~$L_a\varphi$: "Agent a considers φ possible."~~
- $L_a\varphi$: "(*according to the model*), φ is consistent with what a knows ($\neg K_a \neg \varphi$)".

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$\Box\varphi \to \Box\Box\varphi$	Transitive
$\neg\Box\varphi \to \Box\neg\Box\varphi$	Euclidean

S5

The logic **S5** contains the following axioms and rules:

Pc	Axiomatization of Propositional Calculus
K	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$
T	$K\varphi \to \varphi$
4	$K\varphi \to KK\varphi$
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Theorem

S5 *is sound and strongly complete with respect to the class of Kripke frames with equivalence relations.*

Multiagent **S5**

The logic **S5** contains the following axioms and rules:

Pc	Axiomatization of Propositional Calculus
K	$K_i(\varphi \to \psi) \to (K_i\varphi \to K_i\psi)$
T	$K_i\varphi \to \varphi$
4	$K_i\varphi \to K_iK_i\varphi$
5	$\neg K_i\varphi \to K_i\neg K_i\varphi$
MP	$\dfrac{\varphi \qquad \varphi \to \psi}{\psi}$
Nec	$\dfrac{\varphi}{K_i\psi}$

Theorem

Multiagent **S5** *is sound and strongly complete with respect to the class of Kripke frames where each relation is an equivalence relation.*

Truth Axiom

$$K\varphi \rightarrow \varphi$$

Negative Introspection

$$\neg K\varphi \rightarrow K\neg K\varphi$$

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- The agent has not yet entertained possibilities relevant to the truth of φ (the agent is unaware of φ).

Positive Introspection

$$K\varphi \rightarrow KK\varphi$$

The KK Principle

More famous is the “KK principle” (or “positive introspection”):

$$4 \quad K\varphi \to KK\varphi.$$

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Hintikka rejected arguments for 4 based on claims about agents’ introspective powers, or what he called “the myth of the self-illumination of certain mental activities” (67). Instead, his claim was that for a strong notion of knowledge, *knowing that one knows* “differs only in words” from *knowing* (§2.1-2.2).

How Many Modalities?

Fact. In **S5**, there are only three distinct modalities ($\Box$, $\Diamond$, and the "empty modality")