Introduction to Modal Logic

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Propositional Modal Language

Language: Let At be a set of atomic propositions. The set of propositional modal formulas, denoted $\mathcal{L}(\mathsf{At})$, is the smallest set of formulas generated by the following grammar:

$$p \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid \Diamond\varphi$$

where $p \in \mathsf{At}$.

Propositional Modal Language

A formula of Modal Logic is defined *inductively*:

1. Any element of At (called atomic propositions or propositional variables) is a formula
2. $\bot$ is a formula
3. If φ and ψ are formula, then so are $\neg\varphi$ and $\varphi \vee \psi$
4. If φ is a formula, then so is $\Diamond\varphi$
5. Nothing else is a formula

Eg., $\Box(p \to \Diamond q) \vee \Box\Diamond\neg r$; $\neg\Diamond\neg\bot$

Propositional Modal Language

The other Boolean connectives ($\wedge$, $\rightarrow$, and $\leftrightarrow$) are defined as usual

$\top$ is defined as $\neg\bot$.

$\Box\varphi$ is defined as $\neg\Diamond\neg\varphi$

$\Box p \rightarrow p$ is the formula $\neg\neg\Diamond\neg p \vee p$

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Warm-up Question

Can we give a truth-table semantics for the basic modal language?

Hint: there are only 4 truth-functions for a unary operator. Suppose we want $\Box A \to A$ to be valid, but not $A \to \Box A$ and $\neg \Box A$.

Relational Structure

A **relational structure** is a tuple $\langle W, R \rangle$ where $W \neq \emptyset$ and $R \subseteq W \times W$ is a relation.

- Elements of the domain W are called *states*, *possible worlds*, *points*, or *nodes*.
- R is called the *accessibility relation* or the *edge relation*. When wRv we say "w can see v" or "v is accessible from w".
- For $w \in W$, let $R(w) = \{v \mid wRv\}$.

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Warning: Although a relational structure with labels is just a relational structure (with a binary relation and multiple unary relations), they have a specific interpretation in the theory of modal logic.

Examples

- Epistemic models
- Temporal models
- ...

Muddy Children

Three children are outside playing. Two of them get mud on their forehead. They cannot see or feel the mud on their own foreheads, but can see who is dirty.

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Claim: After first question, the children answer "I don't know", after the second question the muddy children answer "I have mud on my forehead!" (but the clean child is still in the dark).

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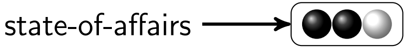

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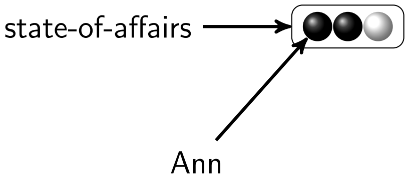

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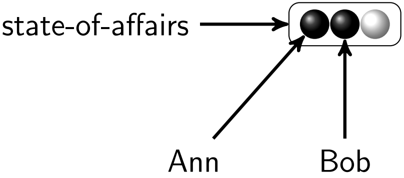

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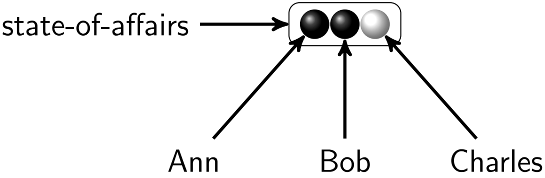

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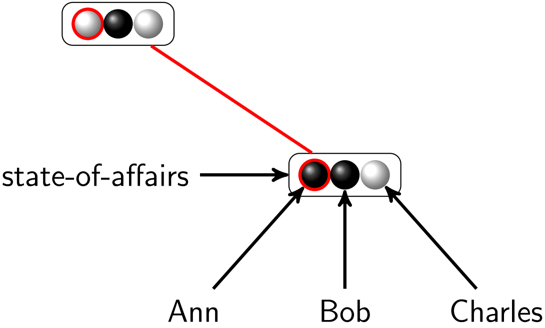

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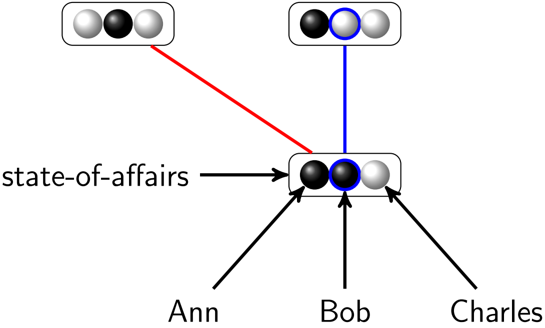

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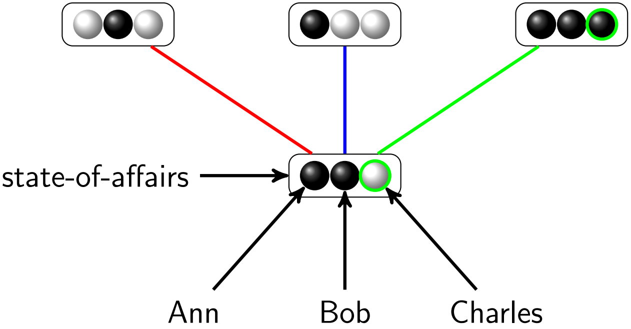

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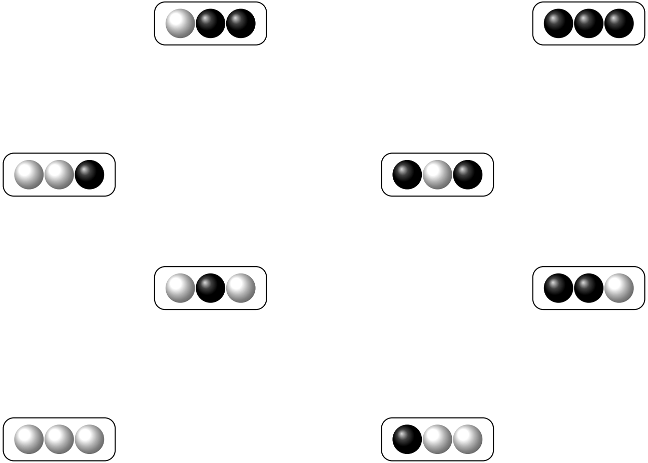
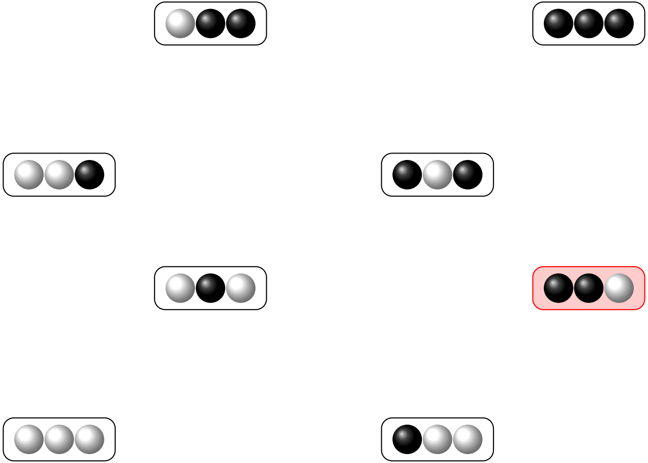


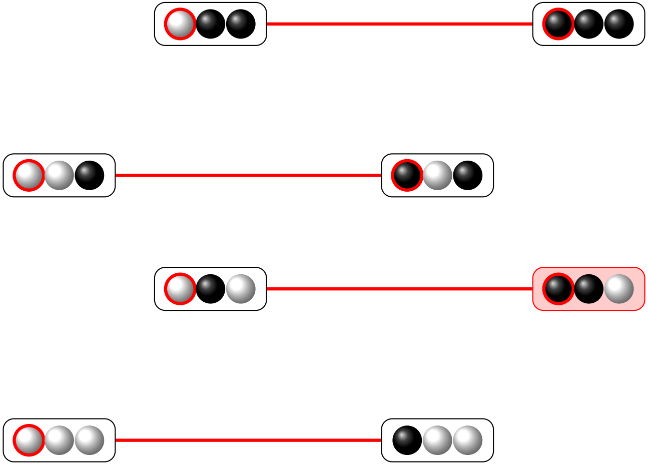
All 8 possible situations

Muddy Children



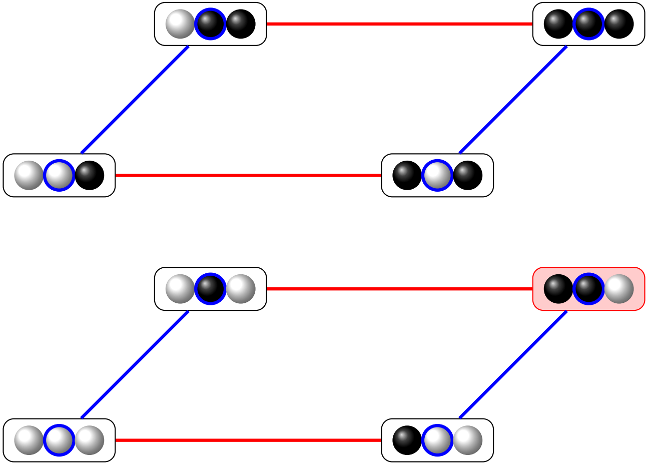
The actual situation

Muddy Children



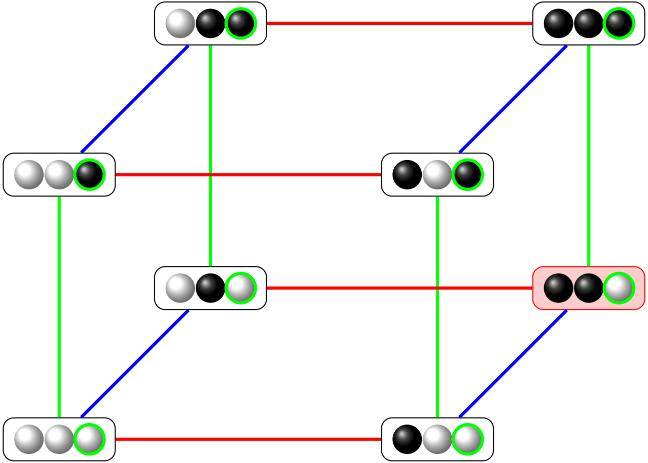
Ann's uncertainty

Muddy Children



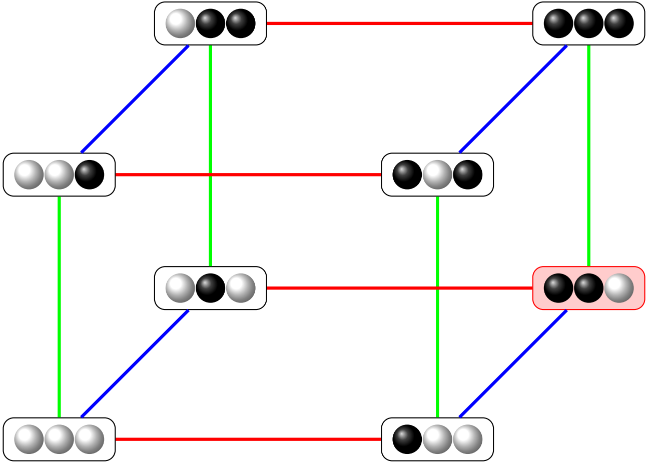
Bob's uncertainty

Muddy Children



Charles' uncertainty

Muddy Children



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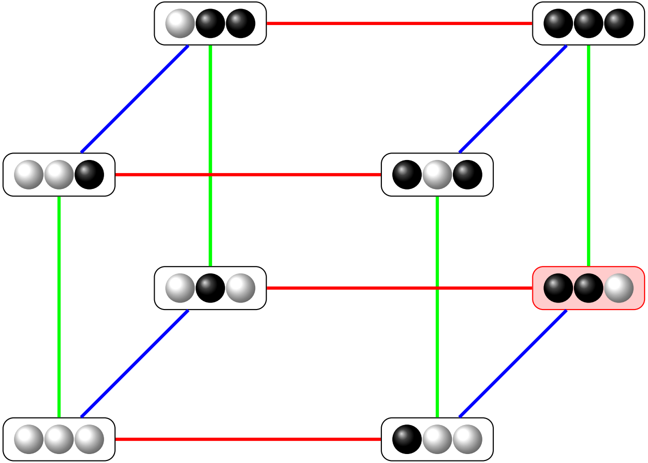
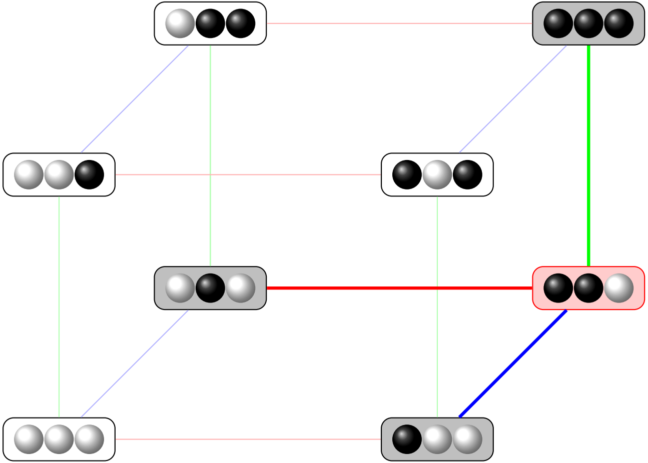


None of the children know if they are muddy

Muddy Children



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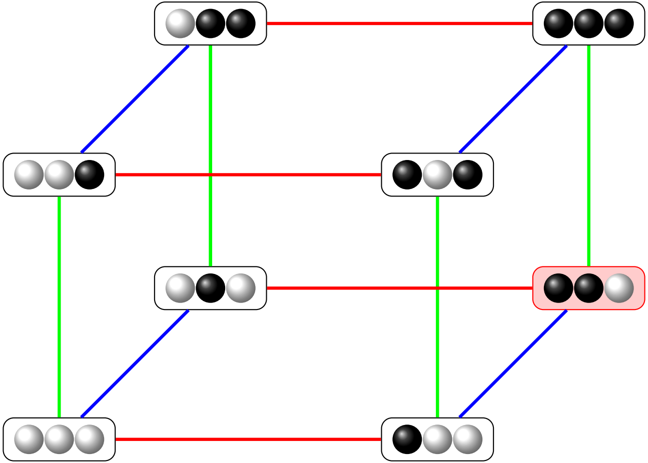
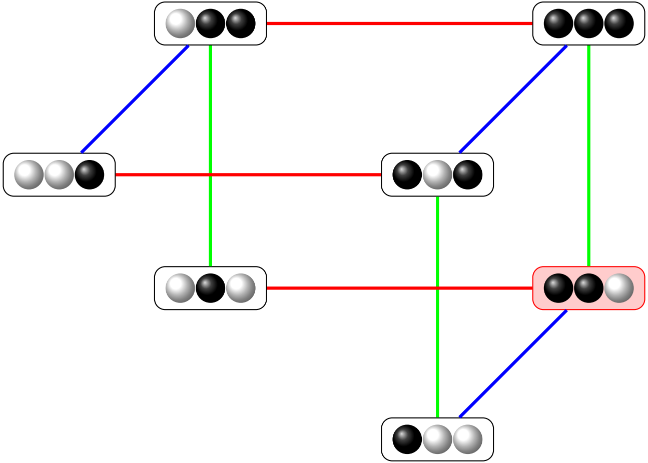


"At least one has mud on their forehead."

Muddy Children



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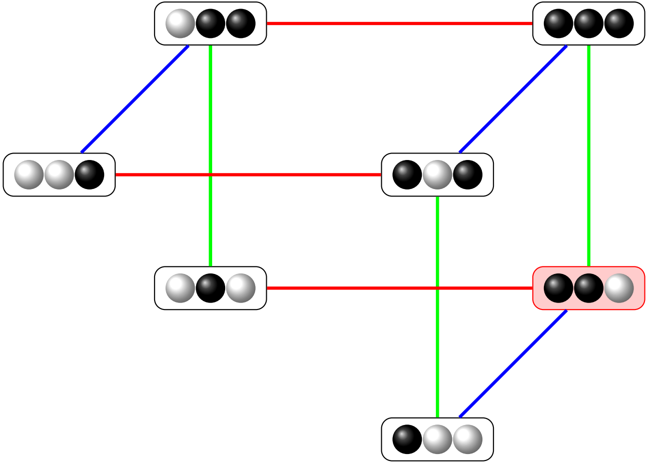

“Who has mud on their forehead?”

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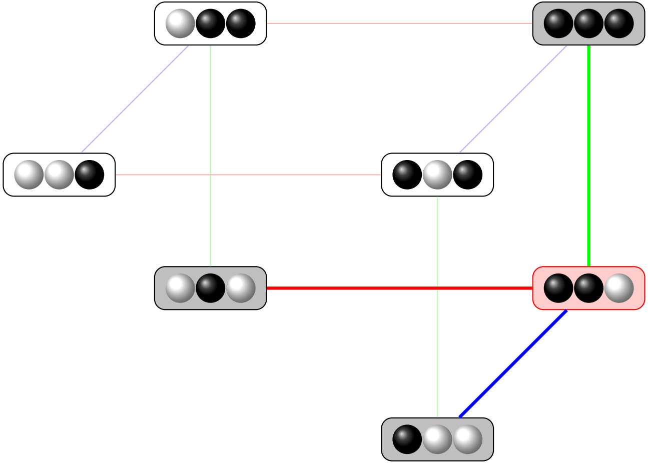

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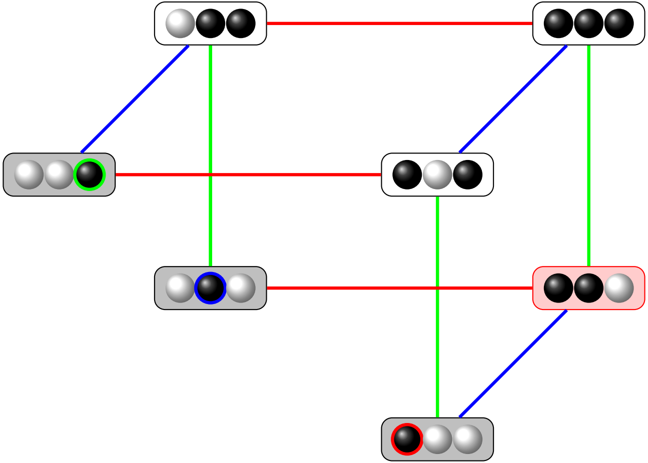
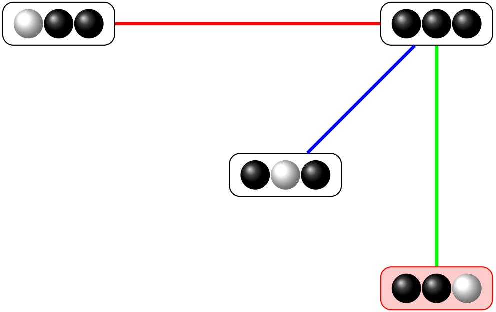


No one steps forward.

Muddy Children



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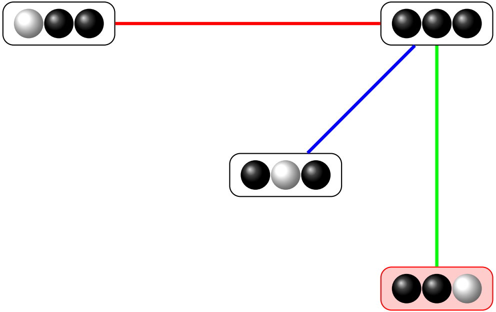

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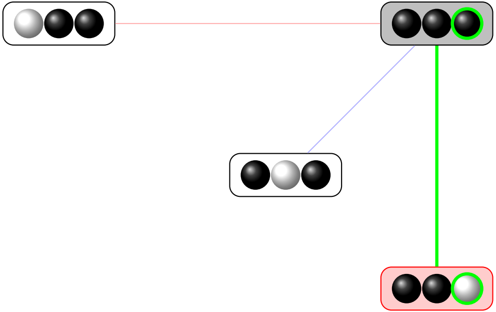

Charles does not know he is clean.

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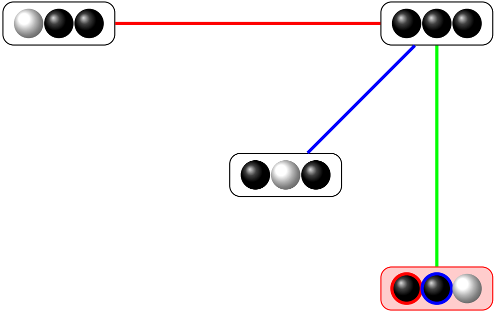

Ann and Bob step forward.

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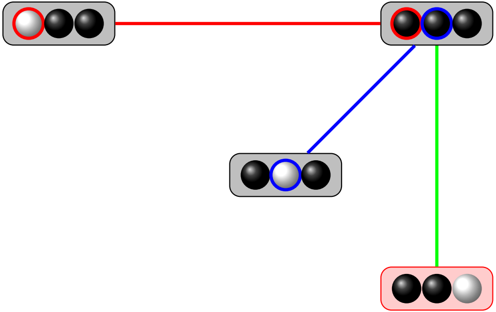

Now, Charles knows he is clean.

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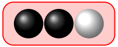

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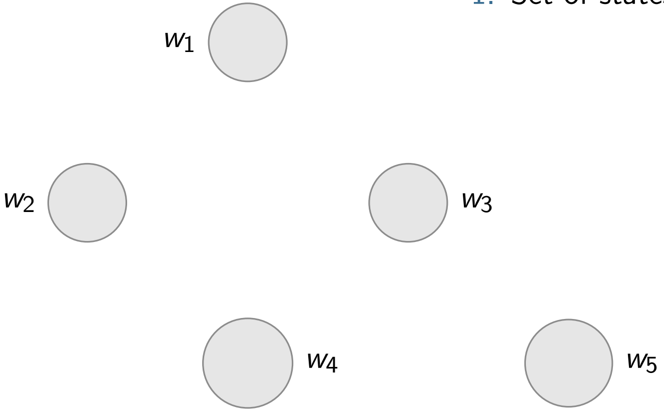
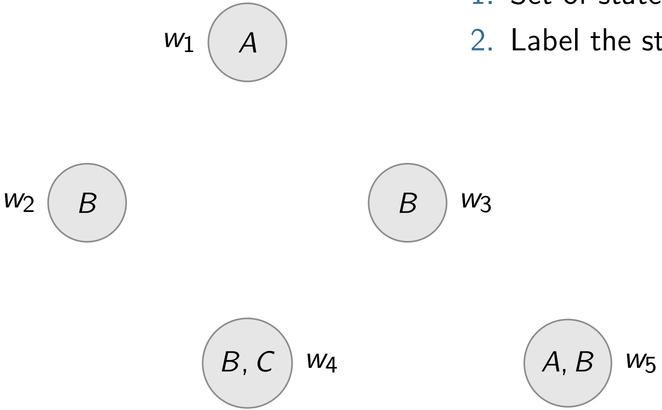
Relational Model



1. Set of states

Relational Model



1. Set of states
2. Label the states

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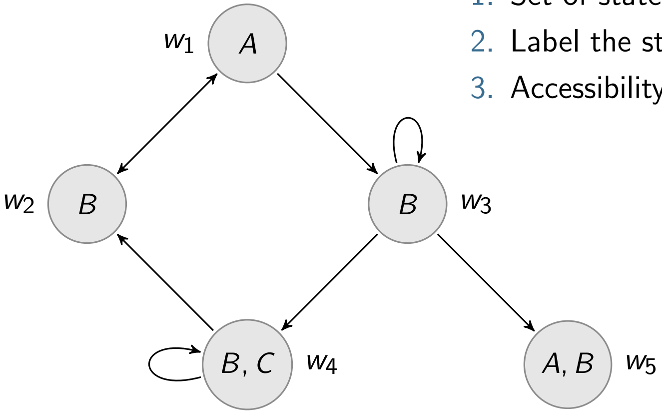

1. Set of states
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3. Accessibility relation

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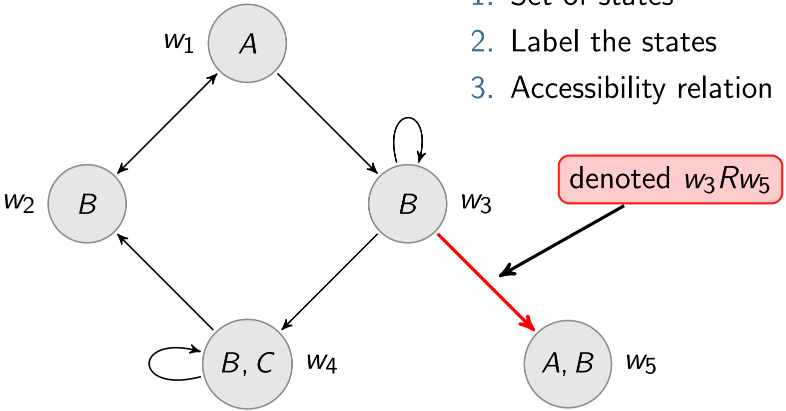

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Frame: $\langle W, R\rangle$, where $W \neq \emptyset$ and $R \subseteq W \times W$

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Model: Suppose that $\mathcal{F} = \langle W, R \rangle$ is a frame. The tuple $\langle W, R, V \rangle$ is a **model based on** $\mathcal{F}$ where $V : \text{At} \to \wp(W)$ is a **valuation function**.

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Pointed Model Suppose that $\mathcal{M} = \langle W, R, V \rangle$ is a model. If $w \in W$, then $(\mathcal{M}, w)$ is called a **pointed model**.

Truth of Modal Formulas

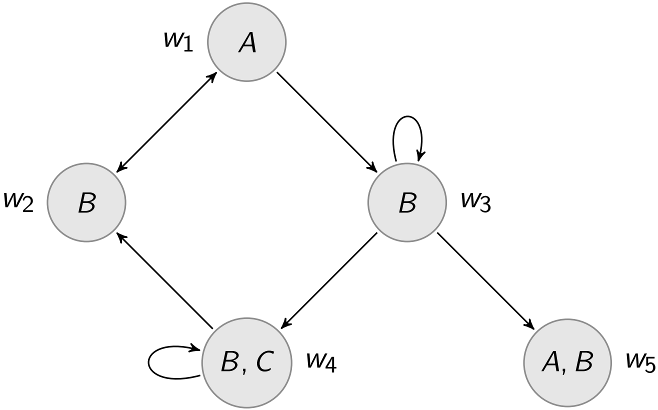
Suppose that $\mathcal{M} = \langle W, R, V \rangle$ is a model. Truth of a modal formula $\varphi \in \mathcal{L}(\mathrm{At})$ at a state w in $\mathcal{M}$, denoted $\mathcal{M}, w \models \varphi$, is defined as follows:

- $\mathcal{M}, w \models p$ iff $w \in V(p)$ (where $p \in \mathrm{At}$)
- $\mathcal{M}, w \not\models \bot$
- $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$
- $\mathcal{M}, w \models \varphi \vee \psi$ iff $\mathcal{M}, w \models \varphi$ or $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \Diamond\varphi$ iff there is a $v \in W$ such that wRv and $\mathcal{M}, v \models \varphi$

Truth of Modal Formulas

- $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \varphi \rightarrow \psi$ iff if $\mathcal{M}, w \models \varphi$, then $\mathcal{M}, w \models \psi$
  iff either $\mathcal{M}, w \not\models \varphi$ or $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \Box\varphi$ iff for all $v \in W$, if wRv then $\mathcal{M}, v \models \varphi$

Example



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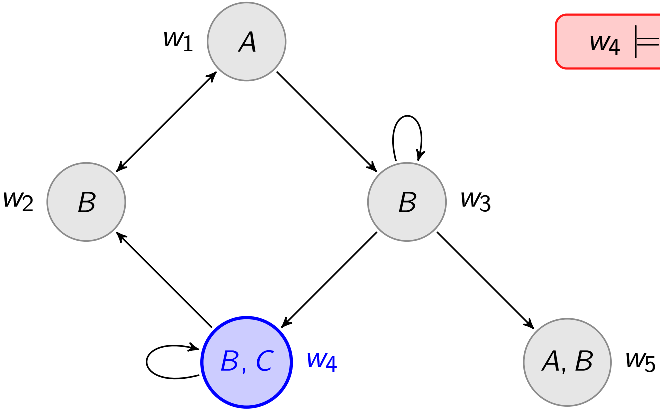
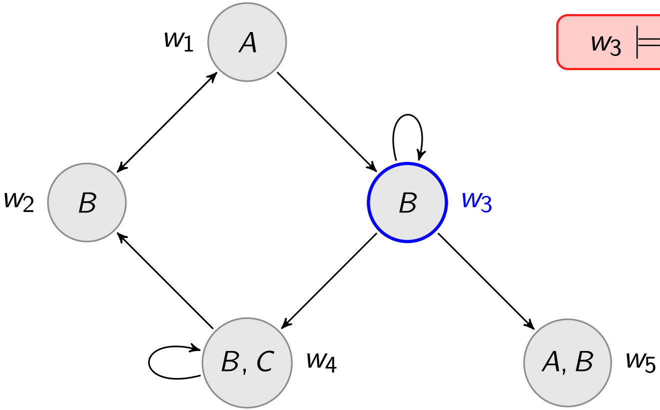


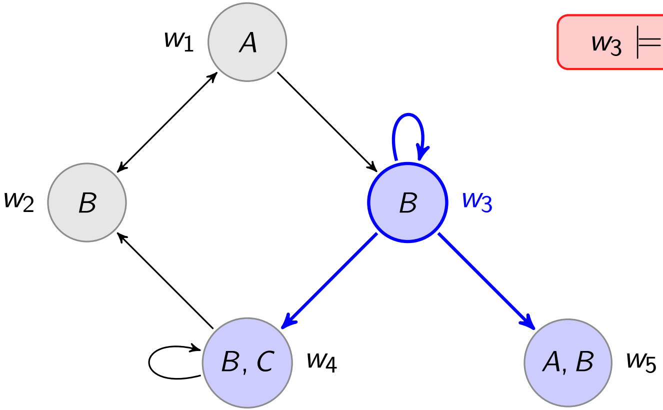
$w_4 \models B \wedge C$

Example



$w_3 \models \Box B$

Example



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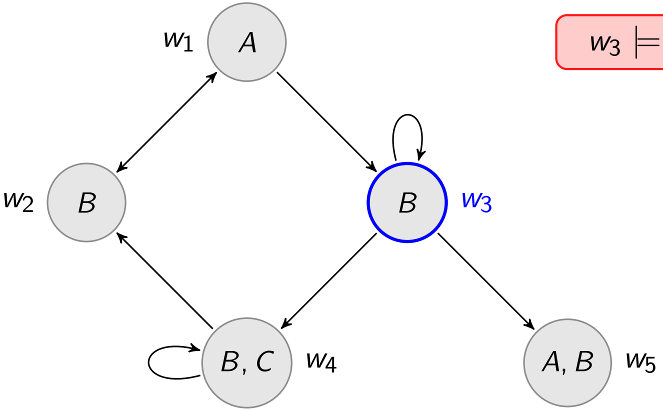

$w_3 \models \Diamond C$

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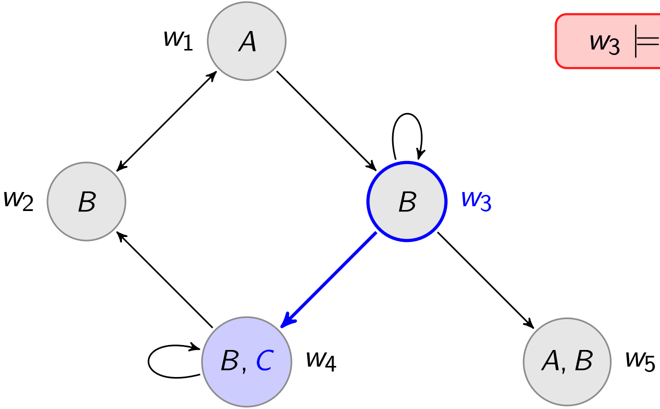

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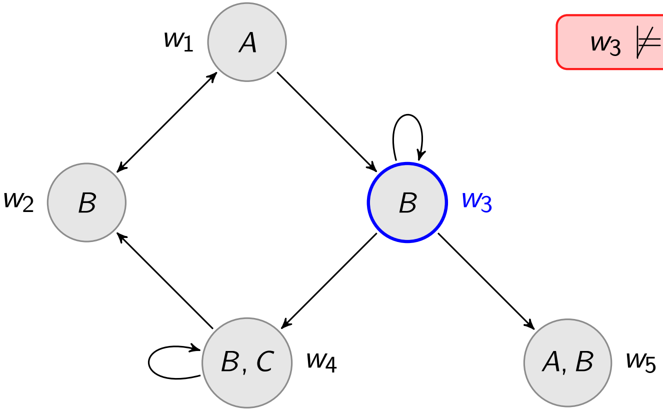

$w_3 \not\models \Box C$

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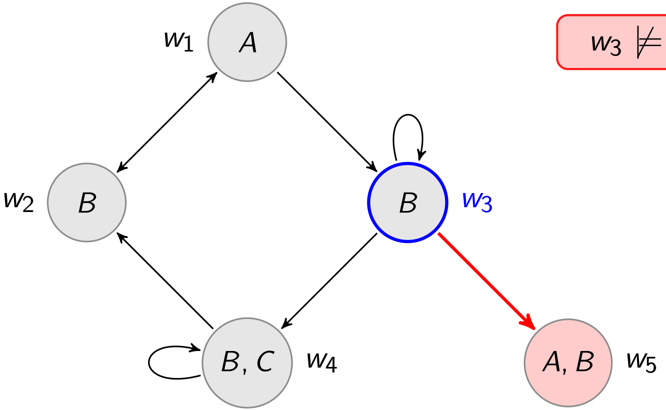

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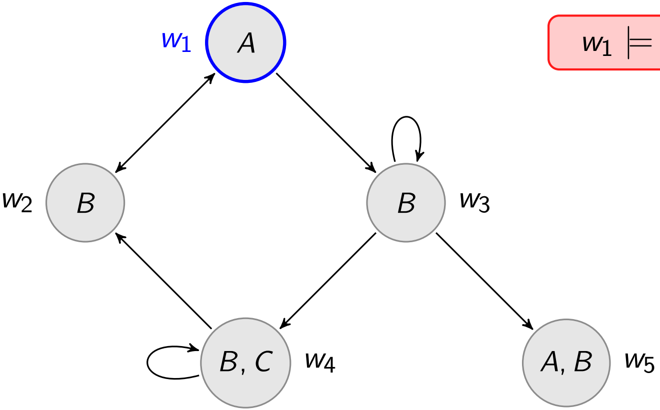
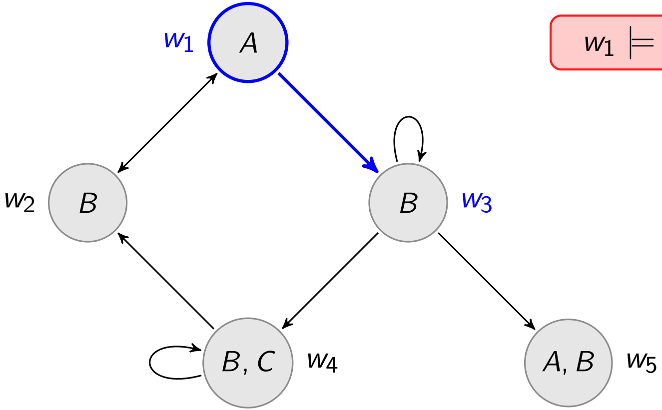


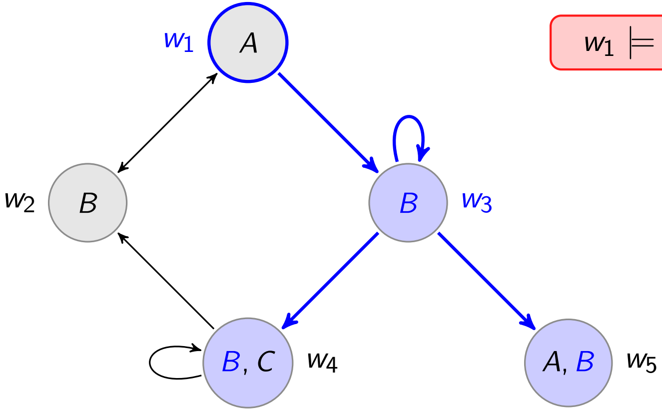
$w_1 \models \Diamond\Box B$

Example



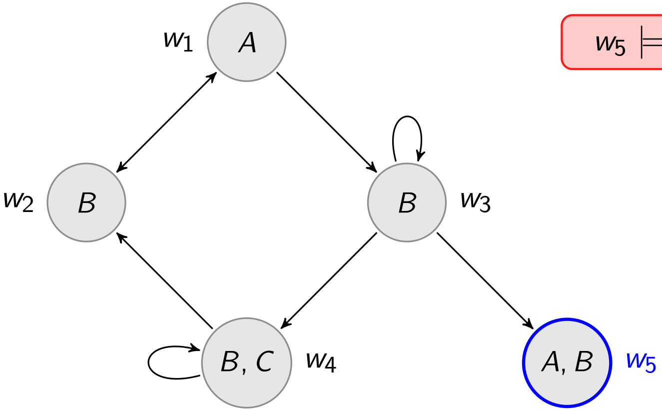
$w_1 \models \Diamond\Box B$

Example



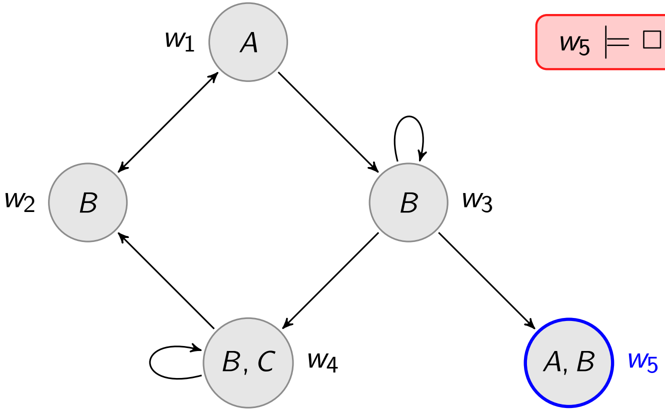
$w_1 \models \Diamond \Box B$

Example



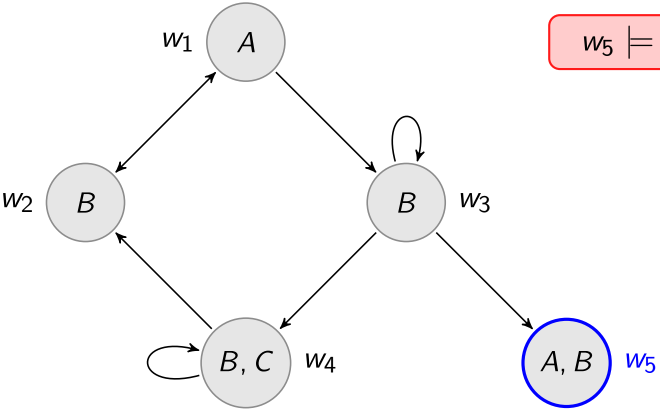
$w_5 \models \Box C$

Example

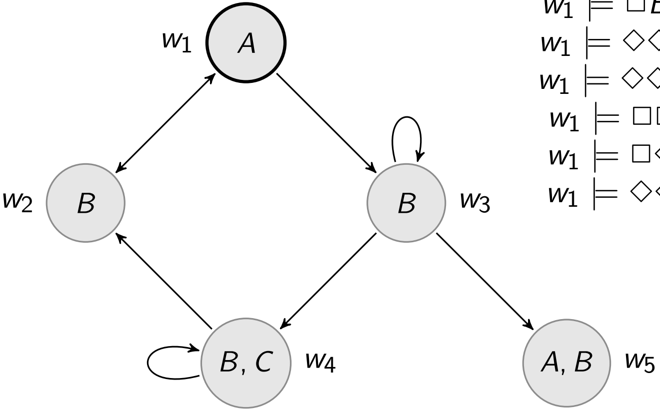


$w_5 \models \Box(B \wedge \neg B)$

Example



$w_5 \models \neg\Diamond B$



$w_1 \models \Box B \wedge B?$

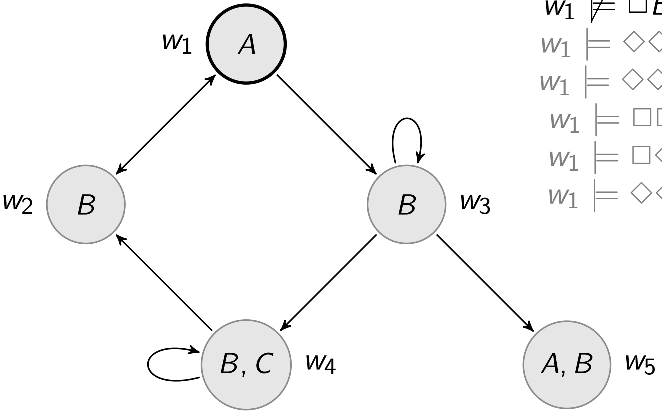
$w_1 \models \Diamond\Diamond B?$

$w_1 \models \Diamond\Diamond\Diamond B?$

$w_1 \models \Box\Box B?$

$w_1 \models \Box\Diamond C?$

$w_1 \models \Diamond\Diamond C?$



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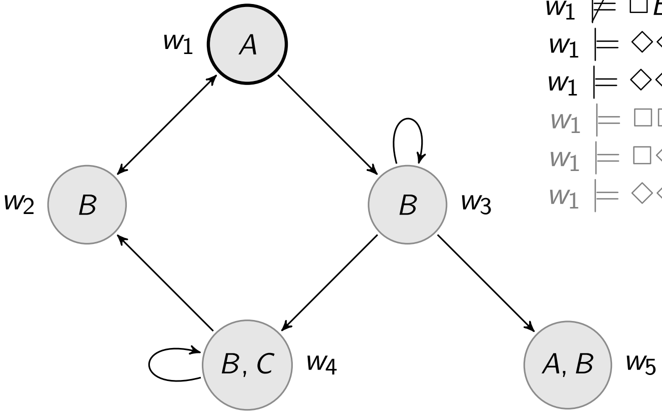

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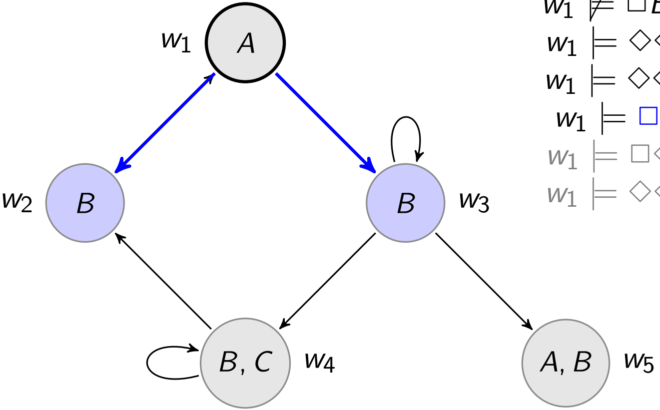

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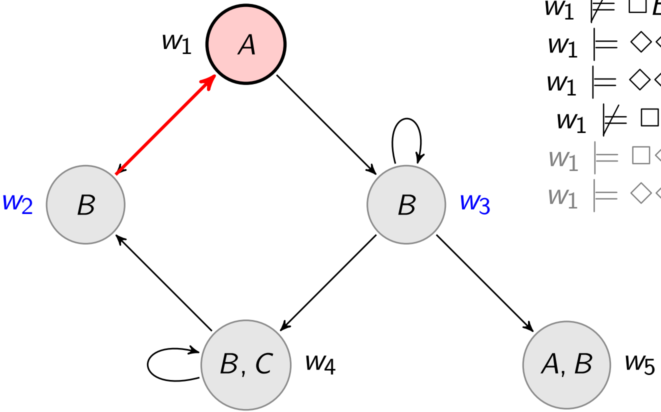

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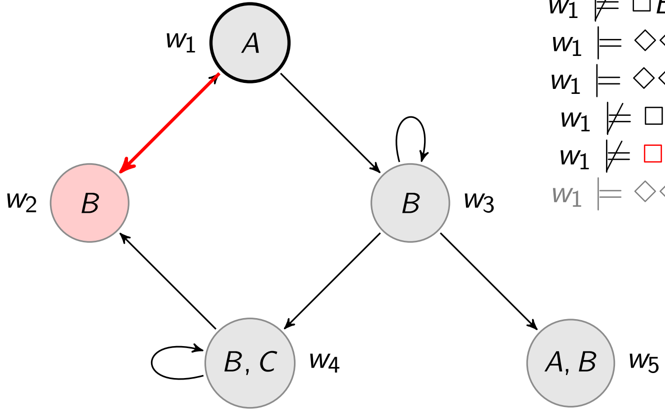

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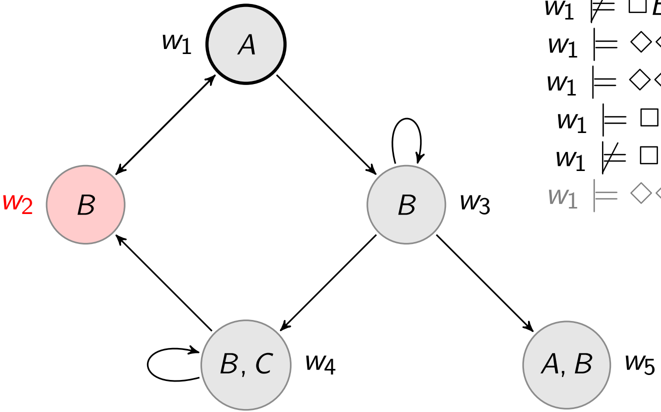

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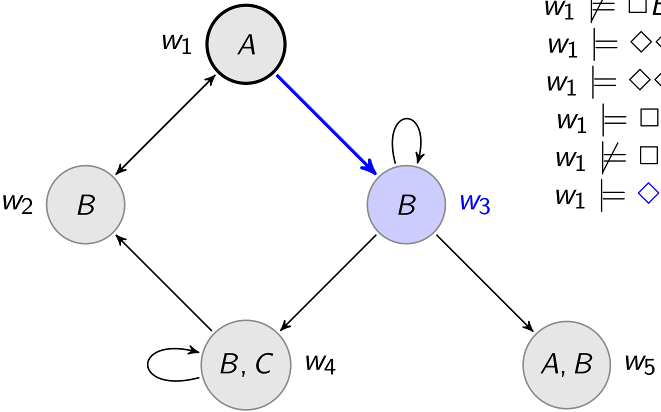

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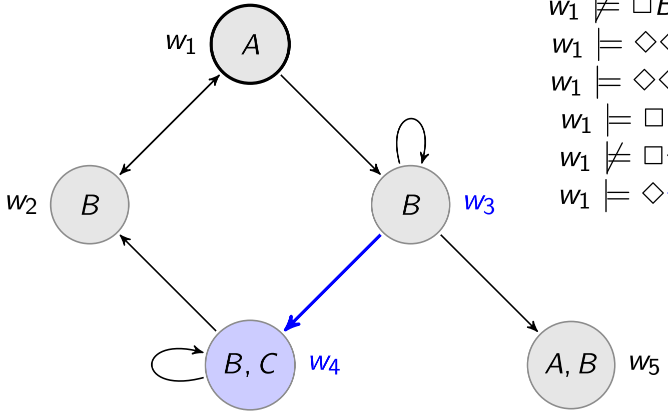

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