Introduction to Modal Logic

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Course Information

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Topics

1. Propositional Modal Logic
2. First-Order Modal Logic
3. Alternative Semantics for Modal Logic (e.g., Topological Models, Neighborhood Models, Algebraic Models, etc.)
4. Applications: (Dynamic) Epistemic Logic, Epistemic Temporal Logic, Logics of Knowledge and Ability

Setting the stage: Classical logic

Propositional Logic (PL)

- Language: $P \wedge Q$, $P \to (Q \vee \neg R)$, etc.
- Proof-Theory: Natural Deduction, Hilbert-style Deductions, Tableaux, etc.
- Semantics: Truth functions

Setting the stage: Classical logic

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First-Order Logic (FOL)

- Language: $x = y$, $\exists x \forall y (P(x) \wedge Q(x, y))$, $\forall x \exists y (F(x) \rightarrow (G(x, y) \wedge \neg R(y)))$, etc.
- Proof-Theory: Natural Deduction, Hilbert-style Deductions, Tableaux, etc.
- Semantics: First-order structures

Notes on propositional and first order logic.

Reasoning with classical logic: pros and cons

Advantages:

- relatively simple syntax and well-understood semantics
- well-developed deductive systems and tools for automated reasoning

Disadvantages:

- cannot adequately represent some aspects natural language
- cannot adequately capture specific modes of reasoning
- *undecidability* of logical consequence and validity (for FOL)

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- Modern modal logic started in the early 1960s with the introduction of relational semantics by Saul Kripke (although see the earlier work by McKinsey and Tarski on logic and topology and Gödel on provability logic).
- There are a wide variety of modal systems, with different interpretations of the modal operators. Modal logic is an important tool in many disciplines: philosophy, computer science, linguistics, economics.

The History of Modal Logic

R. Goldblatt. *Mathematical Modal Logic: A View of its Evolution*. Handbook of the History of Logic, Vol. 7, 2006.

P. Balckburn, M. de Rijke, and Y. Venema. *Modal Logic*. Section 1.7, Cambridge University Press, 2001.

R. Ballarin. *Modern Origins of Modal Logic*. Stanford Encyclopedia of Philosophy, 2010.

What is a modal?

A modality is any word or phrase that can be applied to a statement S to create a new statement that makes an assertion that *qualifies* the truth of S.

Types of Modal Logics

Alethic logic: Necessary and possible truths.

Temporal logic: Temporal reasoning.

Spatial logics: Reasoning about spatial relations.

Epistemic logics: Reasoning about knowledge.

Doxastic logics: Reasoning about beliefs.

Deontic logics: Reasoning about obligations and permissions.

Types of Modal Logics

Logics of multiagent systems: Reasoning about many agents and their knowledge, beliefs, goals, actions, strategies, etc.

Description logics: Reasoning about ontologies.

Logics of programs: Reasoning about program executions.

Game logic: Reasoning about strategies in games.

Provability logic: Reasoning about proofs

Introducing Modal Logic

Modern Modal Logic began with C.I. Lewis' dissatisfaction with the material conditional ($\rightarrow$).

- Irrelevance/non-causality:
  If the Sun is hot, then $2 + 2 = 4$.
- False antecedents:
  If $2 + 2 = 5$ then the Moon is made of cheese.
- Monotonicity:
  If I put sugar in my coffee, then it will taste good. Therefore, if I put sugar and I put oil in my coffee then it will taste good.

Introducing Modal Logic

C.I. Lewis' idea: Interpret 'If A then B' as 'It **must** be the case that A implies B', or 'It is **necessarily** the case that A implies B'

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Judge: $\neg(G \to A)$ $\Leftrightarrow G \wedge \neg A$, therefore G!

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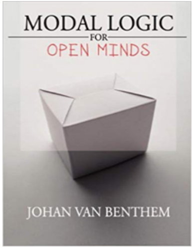
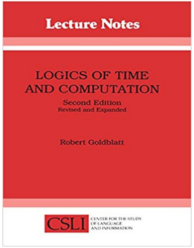
Introducing Modal Logic

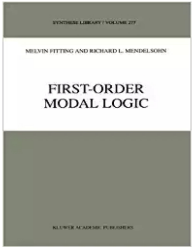
Gradually, the study of the modalities themselves became dominant, with the study of "conditionals" developing into a separate topic.

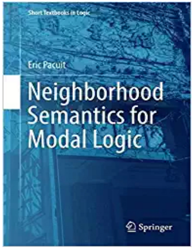
Books

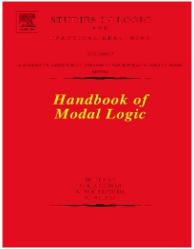

Modal Logic
Patrick Blackburn, Maarten de Rijke and Yde Venema


MODAL LOGIC
FOR
OPEN MINDS
JOHAN VAN BENTHEM

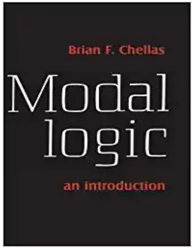

Lecture Notes
LOGICS OF TIME AND COMPUTATION
Second Edition
Revised and Expanded
Robert Goldblatt
CSLI
CENTER FOR THE STUDY OF LANGUAGE AND INFORMATION

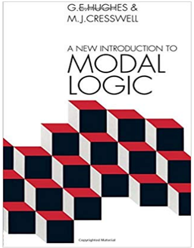

SYNTHESE LIBRARY / VOLUME 277
MELVIN FITTING AND RICHARD L. MENDELSOHN
FIRST-ORDER MODAL LOGIC
KLUWER ACADEMIC PUBLISHERS

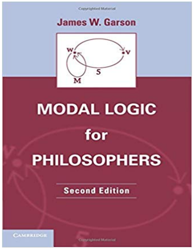

Short Textbooks in Logic
Eric Pacuit
Neighborhood Semantics for Modal Logic
Springer

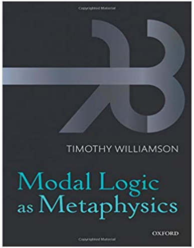

Handbook of Modal Logic

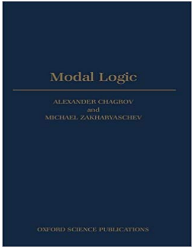
Books


Brian F. Chellas
Modal logic
an introduction


G.E.HUGHES &
M.J.CRESSWELL
A NEW INTRODUCTION TO
MODAL LOGIC


James W. Garson
MODAL LOGIC
for
PHILOSOPHERS
Second Edition
CAMBRIDGE


TIMOTHY WILLIAMSON
Modal Logic
as Metaphysics
OXFORD


Modal Logic
ALEXANDER CHAGROV
and
MICHAEL ZAKHARYASCHEV
OXFORD SCIENCE PUBLICATIONS

Modal Languages

Modal languages extend some logical language (e.g., propositional logic, first-order logic, second-order logic, etc.) with (at least) two new symbols '$\Box$' and '$\Diamond$'.

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$\Diamond\psi$: "it is *possible* that φ is true"

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$\Box\varphi$: "it is *known* that φ is true"

$\Diamond\psi$: "it is *consistent with everything that is known* that φ is true"

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$\Box\varphi$: "it is *will always be* that φ is true"

$\Diamond\psi$: "it is *will sometimes be* that φ is true"

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$\Box\varphi$: "it is *ought to be* that φ is true"

$\Diamond\psi$: "it is *permissible* that φ is true"

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$\Box\varphi$: "it is ______ that φ is true"

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An alternative approach treats modals as *predicates* that apply to terms (that are Gödel numbers of sentences)

J. Stern. *Toward Predicate Approaches to Modality*. Springer, 2016.

Aristotle's Sea Battle Argument

1. If I give the order to attack, then, necessarily, there will be a sea battle tomorrow.
2. If not, then, necessarily, there will not be one.
3. Now, I give the order or I do not.
4. Hence, either it is necessary that there is a sea battle tomorrow or it is necessary that none occurs.

The conclusion is that either it is inevitable that there is a sea battle tomorrow or it is inevitable that there is no battle. So, why should the general bother giving the order?

Aristotle's Sea Battle Argument

There are two possible formalizations of this argument corresponding to different readings of "if A then necessarily B":

$$\begin{array}{l} A \to \Box B \\ \neg A \to \Box \neg B \\ A \vee \neg A \\ \hline \Box B \vee \Box \neg B \end{array} \qquad\qquad \begin{array}{l} \Box(A \to B) \\ \Box(\neg A \to \neg B) \\ A \vee \neg A \\ \hline \Box B \vee \Box \neg B \end{array}$$

Are these two formalizations the same? If not, which argument is valid?

Narrow vs. Wide Scope

"If you do p, you must also do q"

- $p \rightarrow \Box q$
- $\Box(p \rightarrow q)$

Iterations of Modal Operators

$\Box\varphi \rightarrow \Box\Box\varphi$: If I know, do I know that I know?

$\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$: If I don't know, do I know that I don't know?

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What about: $\Diamond\Box\varphi \to \Box\Diamond\varphi$, $\Box\Diamond\varphi \to \Diamond\Box\varphi$, $\varphi \to \Box\Diamond\varphi$, $\Diamond\Box(\varphi \wedge \psi) \to \Diamond\Box\varphi \wedge \Diamond\Box\psi$, $\ldots$?

Propositional Modal Language

Language: Let At be a set of atomic propositions. The set of propositional modal formulas, denoted $\mathcal{L}(\mathsf{At})$, is the smallest set of formulas generated by the following grammar:

$$p \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid (\varphi \wedge \psi) \mid (\varphi \to \psi) \mid \Diamond\varphi \mid \Box\varphi$$

where $p \in \mathsf{At}$.

Notation

- Sometimes we'll use lowercase letters $p, q, r, \ldots$ for atomic propositions and other times we'll use uppercase letters $A, B, C, \ldots$
- The choice of which modal operator is part of the syntax and which is defined is largely conventional. We will use whatever is most convenient.
- When there are multiple modal operators in the language, we will use subscripts $\Box_a$, $\Diamond_a$ or place them "inside" the operators: $[a]$, $\langle a \rangle$

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"This practice is not very consistent, but most readers should agree that it is nice to have different clothes to wear, depending on one's mood"

(van Benthem, pg. 11)

Semantics for Propositional Modal Logic

1. Relational semantics (i.e., Kripke semantics)
2. Neighborhood models
3. Algebraic semantics (BAO: Boolean algebras with operators)
4. Possibility structures
5. Topological semantics (Closure algebras)
6. Category-theoretic (Coalgebras)
7. ...

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Mathematical Background: sets, relations, functions, basic logic, etc.

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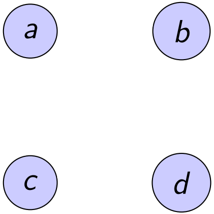

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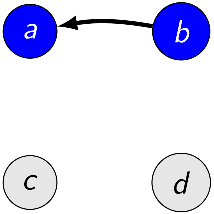

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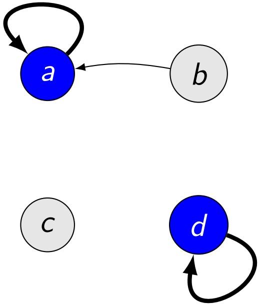

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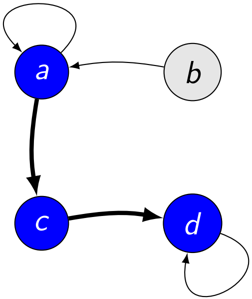

$a \; R \; a$
$b \; R \; a$
$c \; R \; d$
$a \; R \; c$
$d \; R \; d$

Relational Structure

A **relational structure** is a tuple $\langle W, R \rangle$ where $W \neq \emptyset$ and $R \subseteq W \times W$ is a relation.

- Elements of the domain W are called *states*, *possible worlds*, *points*, or *nodes*.
- R is called the *accessibility relation* or the *edge relation*. When wRv we say "w can see v" or "v is accessible from w".
- For $w \in W$, let $R(w) = \{v \mid wRv\}$.

Two generalizations:

1. There is more than one relation
2. The relations can be of arbitrary arity

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Relational structure with labels: $\langle W, R, P_1, P_2, \ldots \rangle$ where $W \neq \emptyset$, R is a (binary or n-ary) relation and for each $k \geq 1$, P_k is unary relation (i.e., $P_k \subseteq W$).

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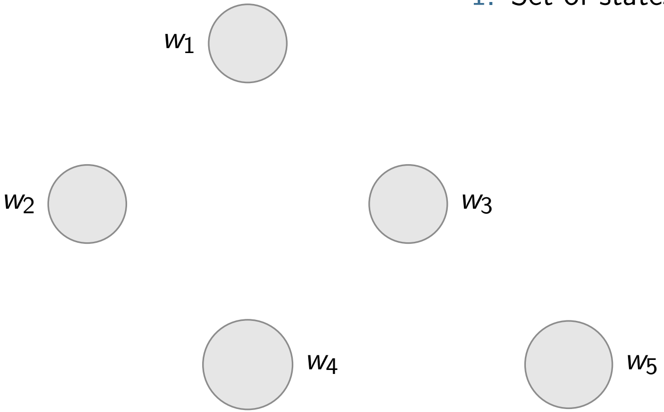
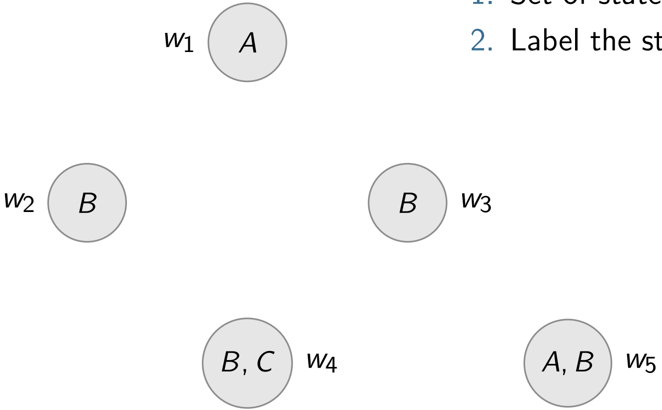
Warning: Although a relational structure with labels is just a relational structure (with a binary relation and multiple unary relations), they have a specific interpretation in the theory of modal logic.

Relational Model



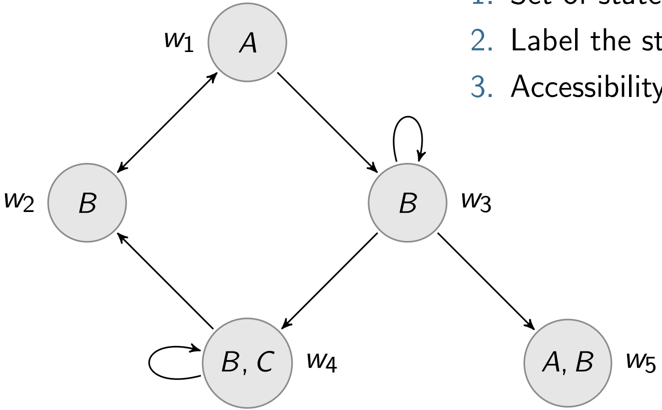
1. Set of states

Relational Model



1. Set of states
2. Label the states

Relational Model



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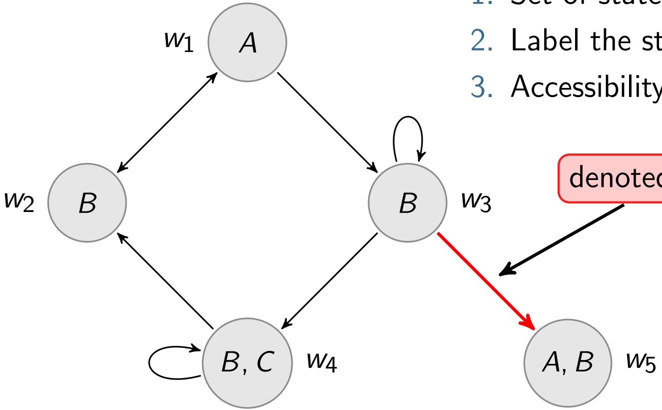
2. Label the states
3. Accessibility relation

Relational Model



1. Set of states
2. Label the states
3. Accessibility relation

denoted $w_3 R w_5$