Frame Definability

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September 21, 2023

1 Definitions

Definition 1 (Frame) A pair $\langle W, R\rangle$ with $W \neq \emptyset$ and $R \subseteq W \times W$ is called a **frame**. Given a frame $\mathcal{F} = \langle W, R\rangle$, a model $\mathcal{M}$ **is based on the frame** $\mathcal{F} = \langle W, R\rangle$ if $\mathcal{M} = \langle W, R, V\rangle$ for some valuation function $V : \mathsf{At} \to \mathcal{P}(W)$. $\triangleleft$

Definition 2 (Frame Validity) Given a frame $\mathcal{F} = \langle W, R\rangle$, a modal formula φ is **valid on** $\mathcal{F}$, denoted $\mathcal{F} \models \varphi$, when for all models $\mathcal{M} = \langle W, R, V\rangle$ based on $\mathcal{F}$, for all $w \in W$, $\mathcal{M}, w \models \varphi$. $\triangleleft$

Definition 3 (Defining a Class of Frames) A modal formula φ **defines the class of frames with property** P provided for all frames $\mathcal{F}$, $\mathcal{F} \models \varphi$ iff $\mathcal{F}$ has property P. In such a case, we say that φ **corresponds** to P. $\triangleleft$

Examples: $\Box\varphi \to \Box\Box\varphi$ corresponds to transitivity; $\Box\varphi \to \varphi$ corresponds to reflexivity; and $\varphi \to \Box\Diamond\varphi$ corresponds to symmetry. (See my notes or the van Benthem book for proofs of these facts.)

2 Digression about Bounded Morphisms

Definition 4 (p-morphism) A **p-morphism** from $\mathcal{F} = \langle W, R\rangle$ to $\mathcal{F}' = \langle W', R'\rangle$ is a function $f : W \to W'$ such that:

- (**forth**) For all $w, v \in W$, wRv implies that $f(w)R'f(v)$
- (**back**) For all $w \in W$, $w' \in W'$, if $f(w)R'w'$, then there is a $v \in W$ such that wRv and $f(v) = w'$.

We say that $\mathcal{F}'$ is a **p-morphic image** of $\mathcal{F}$ if there is a p-morphism from $\mathcal{F}$ *onto* $\mathcal{F}'$ (so the p-morphism is surjective) $\triangleleft$

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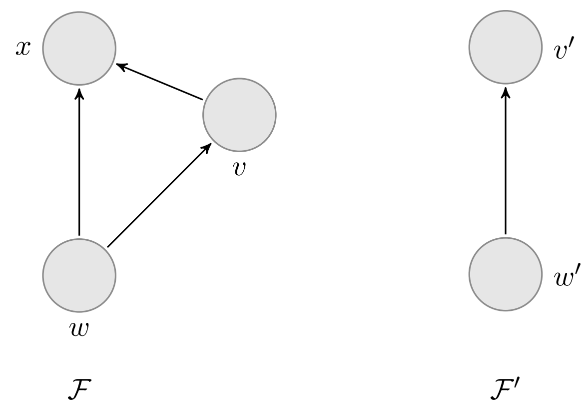
Here are some questions to test your understanding of p-morphisms:

1. Suppose that $\mathcal{F} = \langle W, R\rangle$ and $\mathcal{F}' = \langle W', R'\rangle$ are frames. Prove that $f : W \to W'$ is a p-morphism iff for all $w \in W$,

$$\{f(v) \mid v \in W, wRv\} = \{v' \mid v' \in W', f(w)R'v'\}$$

2. Are there any p-morphisms between these two frames?



3. Suppose that $\mathcal{F} = \langle W, R\rangle$ and $\mathcal{F}' = \langle W', R'\rangle$ are frames and that $\mathcal{F}'$ is a p-morphic image of $\mathcal{F}$. Prove that any modal formula that is valid on $\mathcal{F}$ is valid on $\mathcal{F}'$.

4. Prove that any p-morphic image of a symmetric frame is also symmetric. (Check that the same holds for reflexivity and transitivity.)

3 First Order Logic and Frame Correspondence

This sections assumes familiarity with first order logic. One thing to keep in mind is that we can view a frame $\langle W, R\rangle$ as a first-order structure where the domain is W and R is in the interpretation of a binary predicate symbol (we use "R" for both the binary predicate symbol and the interpretation). From this perspective, we can evaluate whether, for instance, the first-order formula $\forall x\ x\ R\ x$ is true in a frame. A **first-order property** of a frame is any property that is definable by a first-order formula. There are two key questions:

3.1 Does every first-order property of frames have a modal correspondent?

That is, for every first-order property is there is a model formula φ that corresponds to that property? It turns out that there are many examples of first-order properties that are not definable by any modal formula.

Consider the irreflexive property: $\forall x \neg x \, R \, x$. We have the following fact:

Fact 5 *There is no modal formula φ such that for all frames $\mathcal{F}$, we have that $\mathcal{F} \models \varphi$ iff $\mathcal{F}$ is irreflexive.*

Proof. The proof is by contradiction. Suppose that there is a formula φ in a modal langue based on the set At of atomic propositions such that for all frames $\mathcal{F}$, we have that $\mathcal{F} \models \varphi$ iff $\mathcal{F}$ is irreflexive. Consider the following two frames: $\mathcal{F} = \langle \{w_1, w_2\}, \{(w_1, w_2), (w_2, w_1)\} \rangle$ and $\mathcal{F}' = \langle \{w'\}, \{(w', w')\} \rangle$. Since $\mathcal{F}$ is irreflexive, we have that $\mathcal{F} \models \varphi$. We will show that $\mathcal{F}' \models \varphi$. Let $\mathcal{M}'$ be any model based on $\mathcal{F}'$. That is, $\mathcal{M}' = \langle \{w'\}, \{(w', w')\}, V' \rangle$ where $V' : \mathsf{At} \to \wp(\{w'\})$. Consider the model $\mathcal{M} = \langle \{w_1, w_2\}, \{(w_1, w_2), (w_2, w_1)\}, V \rangle$ where for all $p \in \mathsf{At}$,

$$V(p) = \begin{cases} \{w_1, w_2\} & w' \in V'(p) \\ \varnothing & w' \notin V'(p) \end{cases}$$

It is straightforward to check that w_1 and w' are bisimilar, i.e., $\mathcal{M}, w_1 \underline{\leftrightarrow} \mathcal{M}', w'$. Since $\mathcal{M}$ is a model based on $\mathcal{F}$ and $\mathcal{F} \models \varphi$, we have that $\mathcal{M}, w_1 \models \varphi$. Since $\mathcal{M}, w_1 \underline{\leftrightarrow} \mathcal{M}', w'$, we have that $\mathcal{M}', w' \models \varphi$. Since $\mathcal{M}'$ is an arbitrary model based on $\mathcal{F}'$, we have that $\mathcal{F}' \models \varphi$. Since $\mathcal{F}'$ is not irreflexive, this contradicts the assumption that φ corresponds to irreflexivity. QED

Remark 6 *Using the results from the previous section, we can note that $\mathcal{F}'$ is a p-morphic image of $\mathcal{F}$. Then the proof of the above fact follows immediately from the fact that p-morhpic images of frames preserves the validity of modal formulas.*

3.2 Does every modal formula correspond to a first-order property of frames?

There are two standard examples of modal formulas that do not correspond to first-order properties:

- The Gödel-Löb formula $\Box(\Box\varphi \to \varphi) \to \Box\varphi$ corresponds to frames that are transitive and converse well-founded (the latter property is not first-order definable).
- The McKinsey axiom $\Box\Diamond\varphi \to \Diamond\Box\varphi$ does not correspond to a first-order property.